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1

ON THE STEADY TRANSLATION AND REVOLUTION OF A LIQUID SPHERE WITH A SOLID CORE.

BY

SUBODH CHANDRA MITRA.

1. In two important papers[1] Professors Hadamard and Rybezynski discussed the motion of a viscous liquid sphere in an infinite mass of viscous liquid. The methods applied and the results obtained by them were afterwards used by Smoluchowski[2] in order to find the range of validity of Stoke's law of resistance.

The object of the present paper is to investigate, as a variation of the problem of the liquid sphere, the motion of an infinite liquid at rest at infinity due to

(i) the steady translation of a liquid sphere with a solid internal boundary,

(ii) the steady revolution of a liquid sphere with a solid boundary.

It is believed that these cases have not been investigated by any previous writer.

STEADY TRANSLATION.

We shall consider the motion of a viscous liquid sphere of density ρ' and coefficient of viscosity μ' in an infinite mass of liquid of density ρ and coefficient of viscosity μ, the liquid sphere being bounded internally by a concentric solid sphere of radius b and the surface $r=a$ $(a>b)$ separating the viscous liquid sphere from the infinite liquid.

We shall also suppose that the surrounding fluid is free from extraneous forces, while a force $-\frac{K}{\rho'}$ per unit volume acts on the substance of the sphere in the direction of the axis of x.

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[1] Hadamard—Comptes Rendus (1911), p. 1735.
Rybezynski—Bull. Acad. d. Sciences de cracovie (1911), p. 40.

[2] Smoluchowski—On the practical applicability of Stoke's Law of Resistance—Proceedings of the 5th international congress of mathematicians (1912), Vol. 2, p. 192,

Neglecting the inertia terms, the equations of motion of a viscous liquid reduce to the forms

$$\left.\begin{aligned} \mu\nabla^2 u+\rho X &= \frac{\partial p}{\partial x} \\ \mu\nabla^2 v+\rho y &= \frac{\partial p}{\partial y} \\ \mu\nabla^2 w+\rho z &= \frac{\partial p}{\partial z} \end{aligned}\right\} \tag{1}$$

together with the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2}$$

[Lamb, Hydrodynamics, p. 584. Eq. (1) and (2).]

X, Y and Z being the forces parallel to the axes.

Let us first suppose that the sphere is at rest while the liquid has a velocity U parallel to the axis of x at infinity. We shall afterwards impose a velocity $-$ U on both the sphere and the liquid. Then the liquid will be at rest at infinity while the sphere will move with velocity U along the axis of x.

We shall have to consider both the external and the internal motions.

For the external motion we may assume,

$$\left.\begin{aligned} u &= U+\left(B-\frac{Ar^2}{6\mu}\right)\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{2A}{3\mu r} \\ v &= \left(B-\frac{Ar^2}{6\mu}\right)\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) \\ w &= \left(B-\frac{Ar^2}{6\mu}\right)\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) \end{aligned}\right\} \tag{3}$$

These make

$$ux+yv+zw=\left(U-\frac{2B}{r^3}+\frac{A}{\mu r}\right)x \tag{4}$$

The surface-traction components are given by

$$P_{rx} = -\frac{x}{r} p_0 + \left(Ar - \frac{6\mu B}{r} \right) \frac{\partial}{\partial x}\left(\frac{x}{r^3} \right) - \frac{A}{r^2}$$

$$P_{ry} = -\frac{y}{r} p_0 + \left(Ar - \frac{6\mu B}{r} \right) \frac{\partial}{\partial y}\left(\frac{x}{r^3} \right)$$

$$P_{rz} = -\frac{z}{r} p_0 + \left(Ar - \frac{6\mu B}{r} \right) \frac{\partial}{\partial z}\left(\frac{x}{r^3} \right) \qquad (5)$$

[Lamb, Hydrodynamics, p. 584. Eq. (4), (5) and (6)].

For the internal motion let us assume

$$u = \frac{A'}{30\mu'} r^5 \frac{\partial}{\partial x}\left(\frac{x}{r^3} \right) + \frac{A'r^2}{6\mu'} + B' + C' \left[\left(1 - \frac{r^2}{6\mu'} \right) \frac{\partial}{\partial x}\left(\frac{x}{r^3} \right) \right.$$

$$\left. + \frac{2}{3\mu' r} \right] + u'$$

$$v = \frac{A'}{30\mu'} r^5 \frac{\partial}{\partial y}\left(\frac{x}{r^3} \right) + C' \left[\left(1 - \frac{r^2}{6\mu'} \right) \frac{\partial}{\partial y}\left(\frac{x}{r^3} \right) \right] + v'$$

$$w = \frac{A'}{30\mu'} r^5 \frac{\partial}{\partial z}\left(\frac{x}{r^3} \right) + C' \left[\left(1 - \frac{r^2}{6\mu'} \right) \frac{\partial}{\partial z}\left(\frac{x}{r^3} \right) \right] + w' \qquad (6)$$

Where u', v' and w' are such that

$$\nabla^2 u' = 0, \qquad \nabla^2 v' = 0, \qquad \nabla^2 w' = 0 \qquad (7)$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (8)$$

Let us assume

$$\left. \begin{aligned} u' &= D' \frac{\partial}{\partial x}\left(\frac{x}{r^3} \right) \\ v' &= D' \frac{\partial}{\partial y}\left(\frac{x}{r^3} \right) \\ w' &= D' \frac{\partial}{\partial z}\left(\frac{x}{r^3} \right) \end{aligned} \right\} \qquad (9)$$

These values of u', v' and w' will satisfy (7) and (8). Further a comparison of the equations (5) and (14) which is given later will show that these values are consistent with the continuity of the surface tractions at the surface $r=a$.

Thus we can write for the motion inside the liquid sphere,

$$u=\frac{A'}{30\mu'}\,r^5\,\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{A'r^2}{6\mu'}+B'+C'\left[\left(1-\frac{r^2}{6\mu'}\right)\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{2}{3\mu' r}\right]+D'\,\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$v=\frac{A'}{30\mu'}\,r^5\,\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)+C'\left[\left(1-\frac{r^2}{6\mu'}\right)\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]+D'\,\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)$$

$$w=\frac{A'}{30\mu'}\,r^5\,\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)+C'\left[\left(1-\frac{r^2}{6\mu'}\right)\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right]+D'\,\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) \qquad (10)$$

The radial velocity is given by

$$xu+yv+zw=\Bigg[\Bigg(\frac{A'r^2}{10\mu'}+B'-C'\left(\frac{2}{r^3}-\frac{1}{\mu' r}\right)-\frac{2D'}{r^3}\Bigg]x \qquad (11)$$

We find for the pressure,

$$p=-p_0+\left(A'+\frac{C'}{r^3}-K\right)x \qquad (12)$$

Also since

$$\left.\begin{aligned} x^2&=-\tfrac{1}{3}r^5\,\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\tfrac{1}{3}r^2\\ xy&=-\tfrac{1}{3}r^5\,\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\\ xz&=-\tfrac{1}{3}r^5\,\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\end{aligned}\right\} \qquad (13)$$

We find for the surface tractions,

$$P_{rx} = -\frac{x}{r}p_o + \left\{\left(\frac{3}{10}A' - \frac{1}{3}K\right)r^4 + C'\left(r - \frac{6\mu'}{r}\right) - \frac{6\mu' D'}{r}\right\}$$
$$\times \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \left(\frac{1}{3}Kr - \frac{C'}{r^3}\right)$$

$$P_{ry} = -\frac{y}{r}p_o + \left\{\left(\frac{3}{10}A' - \frac{1}{3}K\right)r^4 + C'\left(r - \frac{6\mu'}{r}\right) - \frac{6\mu' D'}{r}\right\}$$
$$\times \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)$$

$$P_{rz} = -\frac{z}{r}p_o + \left\{\left(\frac{3}{10}A' - \frac{1}{3}K\right)r^4 + C'\left(r - \frac{6\mu'}{r}\right) - \frac{6\mu' D'}{r}\right\}$$
$$\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) \qquad (14)$$

Now we may suppose,

(1) there is no tangential slipping over the solid sphere,

(2) there is tangential slipping over the solid sphere.

In both cases the following boundary conditions will hold good.

When $r=a$

(*i*) The radial velocities must vanish.

(*ii*) The velocities are continuous.

(*iii*) The component surface tractions are continuous.

When $r=b$.

(*iv*) The radial velocity due to the motion inside the liquid sphere will vanish.

2. First let us suppose there is no tangential slipping over the solid sphere.

The normal stress is given by the expression,

$$-p_o - 2\left(\frac{3}{10}A' - \frac{K}{3}\right)r - 2C'\left(r - \frac{6\mu'}{r}\right)\frac{x}{r^4} + 12\mu' D'\frac{x}{r^5}$$
$$+\left(\frac{1}{3}Kx - \frac{C'x}{r^3}\right) \qquad (15)$$

the three components of which can be written in virtue of the relation,

$$x p_n = \frac{r^2}{2n+1}\left(\frac{\partial p_n}{\partial x} - r^{2n+1}\frac{\partial}{\partial x}\left(\frac{p_n}{r^{2n+1}}\right)\right) \tag{16}$$

[Lamb, p. 586. Eq. (14)]

$$-p_0\frac{x}{r} + \left\{\frac{2}{10}A'r^4 + C'\left(r - \frac{4\mu'}{r}\right) - \frac{4\mu'D'}{r} - \frac{Kr^4}{3}\right\}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$-\frac{2}{10}A'r - C'\left(\frac{1}{r^2} - \frac{4\mu'}{r^4}\right) + \frac{4\mu'D'}{r^4} + \frac{Kr}{3}\,;$$

$$-p_0\frac{y}{r} + \left\{\frac{2}{10}A'r^4 + C'\left(r - \frac{4\mu'}{r}\right) - \frac{4\mu'D'}{r} - \frac{Kr^4}{3}\right\}\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right);$$

and

$$-p_0\frac{z}{r} + \left\{\frac{2}{10}A'r^4 + C'\left(r - \frac{4\mu'}{r}\right) - \frac{4\mu'D'}{r} - \frac{Kr^4}{3}\right\}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) \tag{17}$$

Subtracting these from (13) we find for the components of tangential stress,

$$\left\{\frac{A'}{10}r^4 - \frac{2\mu'C'}{r} - \frac{2\mu'D'}{r}\right\}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) +$$

$$2\left\{\frac{A'r}{10} - \frac{2\mu'C'}{r^4} - \frac{2\mu'D'}{r^4}\right\};$$

$$\left\{\frac{A'}{10}r^4 - \frac{2\mu'C'}{r} - \frac{2\mu'D'}{r}\right\}\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right);$$

and

$$\left\{\frac{A'}{10}r^4 - \frac{2\mu'C'}{r} - \frac{2\mu'D'}{r}\right\}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) \tag{18}$$

At the surface $r=b$ the radial velocity must vanish and the equations (10) must give the components of tangential velocity. Since we have assumed that there is no tangential slipping at the surface $r=b$, we have the equation

$$\frac{A'}{10}b^4 - \frac{2\mu'C'}{b} - \frac{2\mu'D'}{b} = 0 \tag{19}$$

From the boundary conditions (*i*), (*ii*), (*iii*) and (*iv*) we have the following equations,

$$U-\frac{2B}{a^3}+\frac{A}{\mu a}=0 \tag{20}$$

$$\frac{A'}{10\mu'}a^2+B'-C'\left(\frac{2}{a^3}-\frac{1}{\mu' a}\right)-\frac{2D'}{a^3}=0 \tag{21}$$

$$\frac{A'}{30\mu'}a^5+C'\left(1-\frac{a^2}{6\mu'}\right)+D'=-\frac{Aa^2}{6\mu}+B \tag{22}$$

$$\left(\frac{3}{10}A'-\frac{1}{3}K\right)a^4+C'\left(a-\frac{6\mu'}{a}\right)-\frac{6\mu' D'}{a}=Aa-\frac{6\mu\beta}{a} \tag{23}$$

$$\frac{1}{3}Ka-\frac{C'}{a^3}=-\frac{A}{a^2} \tag{24}$$

$$\frac{A'}{10\mu'}b^2+B'-C'\left(\frac{2}{b^3}-\frac{1}{\mu' b}\right)-\frac{2D'}{b^3}=0 \tag{25}$$

Thus we have seven equations (19), (20), (21), (22), (23), (24) and (25) to determine the seven unknown quantities A, B, A′, B′, C′, D′ and U interms of K.

Solving we have

$$B'=-\frac{1}{10\mu'}\,\frac{a^4+a^3b+a^2b^2+ab^3+b^4}{a^3}\,A'$$

$$C'=\frac{1}{10}\,\frac{b}{a^3}\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)A'$$

$$D'=\frac{1}{20\mu'}\left\{-2\mu'\,\frac{b}{a^3}\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)+b^5\right\}A'$$

and

$$A'=\frac{\dfrac{Ka^5}{3}}{\left\{\dfrac{3a^5}{10}+\dfrac{\mu a^5}{5\mu'}-\dfrac{1}{10}\left(\dfrac{\mu-\mu'}{\mu'}\right)\left(a^4b+a^3b^2+a^2b^3+ab^4+b^5\right)+\dfrac{3}{10}\,\dfrac{\mu-\mu'}{\mu'}\,b^5\right\}} \tag{26}$$

Putting $x=r\cos\theta$, the radial velocity is equal to

$$\left\{\left(\frac{A'r^2}{10\mu'}+B'\right)-C'\left(\frac{2}{r^3}-\frac{1}{\mu' r}\right)-\frac{2D'}{r^3}\right\}\cos\theta \tag{27}$$

The flux $2\pi\psi_1$ through a circle with OX as axis, whose radius subtends an angle θ at O is given by

$$\psi_1=-\frac{1}{2}\left\{\left(\frac{A'r^2}{10\mu'}+B'\right)-C'\left(\frac{2}{r^3}-\frac{1}{\mu' r}\right)-\frac{2D'}{r^3}\right\}r^2\sin^2\theta \tag{28}$$

If we impress on everything a velocity $-U$, we get

$$\psi_1=-\frac{1}{2}\left\{\left(\frac{A'r^2}{10\mu'}+B'\right)-C'\left(\frac{2}{r^3}-\frac{1}{\mu' r}\right)-\frac{2D'}{r^3}-U\right\}\times r^2\sin^2\theta \tag{29}$$

3. Let us now suppose that there is tangential slipping, the coefficient of slipping being denoted by β.

Then exppessing

$$\beta=\frac{\text{Tangential force}}{\text{Relative velocity}}\ \text{(Lamb, p. 572)}$$

we get the following two equations.

$$\frac{A'}{10}b^4-\frac{2\mu' C'}{b}-\frac{2\mu' D'}{b}=\beta\left\{\frac{A'}{30\mu'}b^5+C'\left(1-\frac{b^2}{6\mu'}\right)+D'\right\} \tag{30}$$

and

$$\frac{A'b}{5}-\frac{4\mu' C'}{b^4}-\frac{4\mu' D'}{b^4}=\beta\left\{\frac{A'b^2}{6\mu'}+B'+\frac{2C'}{3\mu' b}\right\} \tag{31}$$

We may take either of the equations (30) or (31) combined with the equations (20), (21), (22), (23), (24) and (25) to determine the seven unknown quantities in terms of K.

STEADY REVOLUTION.

In order to find the steady revolution of a liquid sphere having a solid core, it will be convenient for us to find, first, the revolution of a liquid spheroid bounded internally by a confocal solid spheroid, both rotating about the axis of x, the outer liquid with an angular velocity ω and the inner boundary with an angular velocity ω'. The external motion will be the same as if a rigid ellipsoid of revolution were rotating in an infinite mass of liquid about the axis of x. The external motion due to the revolution of an ellipsoid with three unequal axes has been obtained by D. Edwards (Quart. Journal, Vol. XXVI, 1893). We reproduce below the values of u, v and w, obtained by him.

$$u=\frac{2\sigma p^2 xyz}{(a^2+\lambda)P_\lambda}\left\{\frac{b^2}{b^2+\lambda}-\frac{C^2}{C^2+\lambda}\right\} \tag{32}$$

$$v=-\sigma\left(b^2B_\lambda+c^2C_\lambda\right)z+\frac{2\sigma p^2y^2z}{(b^2+\lambda)P_\lambda}\left(\frac{b^2}{b^2+\lambda}-\frac{c^2}{c^2+\lambda}\right) \tag{33}$$

$$w=\sigma\left(b^2B_\lambda+c^2C_\lambda\right)y+\frac{2\sigma p^2yz^2}{(c^2+\lambda)P_\lambda}\left(\frac{b^2}{b^2+\lambda}-\frac{c^2}{c^2+\lambda}\right) \tag{34}$$

and

$$\sigma=\frac{w}{b^2B+c^2C} \tag{35}$$

$$A_\lambda=\int_\lambda^\infty\frac{d\lambda}{(a^2+\lambda)P_\lambda},\quad B_\lambda=\int_\lambda^\infty\frac{d\lambda}{(b^2+\lambda)P_\lambda},\quad C_\lambda=\int_\lambda^\infty\frac{d\lambda}{(c^2+\lambda)P_\lambda} \tag{36}$$

$$P_\lambda=\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}} \tag{37}$$

$$\frac{1}{p^2}=\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2} \tag{38}$$

a, b, c being the semi-axes of the ellipsoid.

Now let us suppose that the boundary of the innermost spheroid is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1 \tag{39}$$

While that of the outer spheroid is given by

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2+z^2}{b^2+\lambda_1}=1 \tag{40}$$

It is obvious that λ_1 is a positive quantity. Then for any point outside the outer spheroid, u, v and w are give by (34) where we put $b=c$.

For any point on the outer spheroid we must replace λ by λ_1 $(\lambda>\lambda_1)$

Then

$$\sigma=\frac{w}{2b^2 B_{\lambda_1}} \tag{41}$$

$$\left.\begin{aligned} u&=0\\ v&=-\omega\frac{B_\lambda}{B_{\lambda_1}}z\\ w&=\omega\frac{B_\lambda}{B_{\lambda_1}}y \end{aligned}\right\} \tag{42}$$

The normal velocity is clearly zero.

The component surface tractions are given by

$$P_{\lambda x}=-p_0\frac{px}{a^2+\lambda}$$

$$P_{\lambda y}=-p_0\frac{py}{b^2+\lambda}+\frac{2w\mu pz}{B_{\lambda_1}(b^2+\lambda)^2(a^2+\lambda)^{\frac{1}{2}}}$$

$$P_{\lambda z}=-p_0\frac{pz}{b^2+\lambda}-\frac{2w\mu py}{B_{\lambda_1}(b^2+\lambda)^2(a^2+\lambda)^{\frac{1}{2}}} \tag{43}$$

Since the condition of finitenees at the origin is no longer imposed, we may assume for the internal motion,

$$\left.\begin{aligned} u&=0\\ v&=-Az-CB_\lambda z\\ w&=Ay+CB_\lambda y \end{aligned}\right\} \tag{44}$$

These satisfy the equations of continuity and the equations of motion.

Also

$$\left.\begin{aligned} P_{\lambda x} &= -p_0 \frac{px}{a^2+\lambda} \\ P_{\lambda y} &= -p_0 \frac{py}{b^2+\lambda} + \frac{2\mu' Cpz}{(b^2+\lambda)^2(a^2+\lambda)^{\frac{1}{2}}} \\ P_{\lambda z} &= -p_0 \frac{pz}{b^2+\lambda} - \frac{2\mu' Cpy}{(b^2+\lambda)^2(a^2+\lambda)^{\frac{1}{2}}} \end{aligned}\right\} \tag{45}$$

The boundary conditions are when $\lambda=\lambda_1$ $u=o$, $v=-\omega z$ and $\omega=\omega y$ (46)

When $\lambda=0$

$$u=o,\ v=-\omega' z \text{ and } w-\omega' y. \tag{47}$$

The component surface tractions must be continuous when $\lambda=\lambda_1$.

These give the following equations:

$$A+CB_{\lambda_1}=\omega \tag{48}$$

$$A+CB=\omega' \tag{49}$$

and

$$\frac{\omega\mu}{B_{\lambda_1}}=c\mu' \tag{50}$$

Solving we have

$$A=\frac{\omega(\mu'-\mu)}{\mu'} \tag{51}$$

$$C=\frac{\omega\mu}{B_{\lambda_1}\mu} \tag{52}$$

and

$$\omega'=\frac{\omega(\mu'-\mu)}{\mu'}+\frac{\omega\mu B}{B_{\lambda_1}\mu'} \tag{53}$$

Thus we get A, C and ω'.

In the case of the sphere we have $a=b=c$.

Therefore $$B_{\lambda_1}=\frac{2}{3}\,\frac{1}{(a+\lambda_1)^3}=\frac{2}{b^3}\,\frac{1}{3} \qquad (54)$$

where we write $$(a+\lambda_1)=b \qquad (55)$$

and $$B=\frac{2}{3}\,\frac{1}{a^3} \qquad (56)$$

Therefore

$$\omega'=\frac{\omega(\mu'-\mu)}{\mu'}+\frac{3}{2}\,\frac{\omega\mu}{\mu'}\,\frac{b^3}{a^3} \qquad (57)$$

In conclusion, I wish to express my indebtedness to Dr. N. M. Basu for his valuable criticism and help in the preparation of the paper.

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2

ON THE SECONDARY SPECTRUM OF HYDROGEN.

By

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Although the application by Bohr and Sommerfeld, of Quantum-theory to spectroscopy, has met with a phenomenal success, the only atoms, of which the line-spectra, have as yet been quantitatively accounted for are those of hydrogen and ionised helium. The next-structure in the order of complexity is the helium atom. But as it is a problem of three bodies, no exact solutions have been obtained. The hydrogen molecule is still more complex, as it consists of two electrons and two hydrogen-nuclei. The only existing model of the hydrogen-molecule is that of Debye, but the dynamical solution of the Debye-model has not been effected yet, although it has attracted the attention of Silberstein and Saha. It is practically established now that the secondary spectrum of hydrogen is emitted by the hydrogen-molecule. So a theory of the secondary spectrum must rest on a workable model of the molecule. The present paper embodies such a model and the Hamilton-Jacobian equation of the same can be solved to a certain order of approximation. A frequency-formula has been deduced; but as the secundary spectrum of hydrogen consists of an extremely large number of lines, it is idle to identify each of these lines with some lines calculated from formula, at the present state of the subject. But as Sommerfeld has pointed out, some other features of the secondary spectrum should be studied with a view to observing some regularity, and if such is observed, any new theory should be based on them. Glitscher has observed[1] that if the maxima in the energy-diagram of the secondary spectrum are superposed on Balmer lines, then each Balmer line nearly coincides with a corresponding maximum. The model suggested here appears to account for this to some extent.

In order to arrive at the proposed model we start from the genesis of the molecule. When two neutral hydrogen atoms approach each other, the centre of gravity of their nuclei is approximately at rest or in uniform motion, since the masses of the electrons are small compared with those of the nuclei. Now since the mutual force between the nuclei is one of repulsion they would fly apart after some time unless some electrons intervene. Let us place one electron at the centre of gravity of these two nuclei, so that this electron moves in a practically force-free field. The two nuclei will now describe closed orbits about this electron, and these would form an ellipse-verein or degenerate into a circle. The remaining electron may now describe some orbit at a comparatively large distance from this complex structure. This completes the model.

The potential of this complex structure consisting of an electron and two nuclei can be shown to be approximately that of two centres[1] of force, Or we may replace the two revolving nuclei by a ring of electric charge and expand its potential in a series form. But it is immaterial which way we regard the problem, as it is the form of the series-formula that concerns us and not the numerical value of the constants involved therein. In accordance with the latter view the potential is of the form,

$$V = \frac{e}{r} + \frac{c_1}{r^3} + \frac{c_2}{r^5} + \qquad \ldots \quad 1$$

If we retain terms up to $1/r^3$ only and quantise the atom in the polar coordinates of the valency-electron, we shall then arrive at the usual Rydberg Farmula :—

$$\gamma = N \left| \frac{1}{(n+\alpha)^2} - \frac{1}{(p+\beta)^2} \right|$$

Now the term c_1, is proportional to the square of the radius of the ring of nuclei. If we assume that this nuclear ring also is quantised, then c_1 becomes a function of these quantum numbers. Thus the terms α, β in Rydbergs formula quoted above, are also functions of these quantum-numbers. Assuming that these nuclei, each of mass M, describe the same circle with radius a, and have an angular velocity ω, the equation of motion is

$$Ma\omega^2 = \frac{3e^2}{4a^2} ,$$

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[1] *Cf.* Tank. Ann. d. Physik, 1919.

subject to the quantum-condition,

$$2Ma^2\omega = \frac{n_0 h}{2\pi},$$

Hence
$$a = \frac{n_0^2 h^2}{12\pi^2 M e^2}$$

It is show in Sommerfelds'. Atom-bau, 2nd *ed.* that the constant a in Rydberg's formula is proportional to c_1 and therefore to n_0^4.

We now introduce the idea that when a quantum-radiation takes place, both the numbers n and n_0 suffer a quantum-transit, which amounts to saying that a re-arangement of both the inner structure and and the outer electrons takes place during a radiation. Under the circumstances, the frequency-formula takes the form,

$$\gamma = \frac{N_0}{(n+k.n_0^4)^2} - \frac{N_0}{(p+k.p_0^4)^2}$$

where k is a small quantity of the order of m/M.

This can be approximately written as follows :—

$$\gamma = N_0\left(\frac{1}{n^2} - \frac{1}{p^2}\right) - N_0.2k.\left(\frac{n_0^4}{n^3} - \frac{p_0^4}{p^3}\right).$$

From the expression for the radius of the nuclear ring, it is evident that unless n_0 is very large, this ring is much smaller than the one-quantum or two-quantum orbit of the outer electron. It is thus obvious that corresponding to every quantum-jump of the outer electron, quite a large number of transitions in the value of n_0 is possible, and this will give rise to a large number of closely grouped lines. This accounts for the many-lined character of the secondary spectrum. Also the form of the frequency-formula at once visualises a close relation between the secondary-spectrum and the Balmer lines. For instance, if we put $n=2$ and $p=3$, this corresponds to a quantum-transit in the hydrogen *atom* which gives rise to the line Hα. In the molecule however we shall get large cluster of lines somewhere near Hα. Similarly, if we put $n=2$ and $p=4$, we shall get another cluster of lines in the neighbourhood of H$_\beta$ and so on. It is to be borne in mind that these clusters most probably overlap each other so that no well-defined line of demarkation exists between them. The net appearance of this theoretical spectrum is neither that of bands nor of series-lines. The puzzling character of the secondary spectrum and its non-conformity to any class is quite well-known.

Of course an actual calculation of these lines and their identification with the individual lines in the secondary spectrum will lead to nowhere, but there are other features of the formula which lend themselves easily to an experimental test.

As already mentioned in the introduction, Glitscher observed that if the maxima in the energy-diagram of the secondary spectrum of hydrogen were denoted by the symbol H_α', H_β', etc., then the difference in wave-number between the above and the Balmer-lines; *i.e.*, the quantities $H_\alpha'-H\alpha$, $H_\beta'-B_\beta$, etc., were approximately constant.

We can reasonably assume that the state of the inner core of the molecule which corresponds to these maxima, must be the most probable states. Let $n_0=p_0$ be the quantum-number denoting these most probable states. Then it is easily seen that

$$H_\alpha'-H\alpha= -N_0\cdot 2k\cdot n_0^4 \left(\frac{1}{2^3}-\frac{1}{3^3}\right)$$

$$H_\beta'-H_\beta= -N_0.2k.n_0^4 \left(\frac{1}{2^3}-\frac{1}{4^3}\right)$$

$$H_\gamma'-H_\gamma= -N_0.2k.n_0^4 \left(\frac{1}{2^3}-\frac{1}{5^3}\right), \text{ etc.}$$

Hence the ratios of these are

$$=\frac{1}{2^3}-\frac{1}{3^3}:\frac{1}{2^3}-\frac{1}{4^3}:\frac{1}{2^3}-\frac{1}{5^3}: \ . \ . \ .$$

$$=0{\cdot}73 : 0{\cdot}91 : 0{\cdot}97 : 1 : \ . \ . \ .$$

On the other hand from the energy-diagram we find :—

$$H\alpha'-H\alpha : H_\alpha'-H_\beta : H_\gamma'-H_\gamma : \ . \ . \ .$$

$$=1380 : 1470 : 1542 : 1652 : \ . \ . \ . \quad =0{\cdot}84 : 0{\cdot}89 : 0{\cdot}95 : 1 : \ . \ . \ .$$

If the agreement between the theoretical and observed values is not quite close, these values are at least of the same order of magnitude. We must make room for the possibility that n_0 may not equal p_0 in a quantum jump corresponding to any maximum in the energy-diagram but may be slightly different from one another; this will certainly modify the value of the ratios to some extent.

It is natural to expect that the complex core of the molecule is not very stable. It is thus obious that at low voltages, the spark or the arc spectrum should consist of the secondary lines emitted by the molecule; and when the voltage is increased, the molecule atomises and the Balmer lines should make their appearance. This is well corroborated by experimental results.

There are reasons to believe that the resonance and ionisation-potentials determined by electron-collision (Horton and Davies) are not really those of the atom, but of the molecule (see Foote and Mohler's "Origin of Spectra," p. 75). If we put $n=1$ instead $n=2$, as in Lyman's series, then the ionisation and resonance-potentials of our molecule will differ from those of the atom by a quantity of the order m/M, Hence our model bears out the conjecture of Foote and Mohler in a striking manner.

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3

HIGHER ORDER TIDES IN CANALS OF VARIABLE SECTION.

BY

NRIPENDRA NATH SEN, D.SC.

The only problem of higher order tides completely solved is the one considered by Airy[1] and McCowan[2] in which the section is uniformly rectangular. The object of the present paper is (1) to establish the exact equations for free tidal oscillations in canals of variable section and (2) to determine higher order tides in a parabalic canal.[3]

2. Taking the origin on the undisturbed level and the axis of x parallel to the length of the canal, the exact equations for free tidal oscillations in canals of variable section may easily be seen to be

$$\frac{\partial \eta}{\partial t}+\frac{1}{b}\frac{\partial}{\partial x}\, b\,(h+\eta)\,u=0 \qquad \ldots \quad (1)$$

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}=-g\,\frac{\partial \eta}{\partial x} \qquad \ldots \quad (2)$$

where u, η, b, h, p are velocity, tidal elevation, breadth, depth and pressure at a distance x.

3. Let $b=b_0$, $h=h_0\left(1-\frac{x^2}{a^2}\right)$, so that the longitudinal section is a parabola. In this case, we have from (1) and (2), after a little simplification.

$$\frac{\partial^2 \eta}{\partial t^2}-\frac{gh_0}{a^2}\,\frac{\partial}{\partial z}\,(1-z^2)\,\frac{\partial \eta}{\partial z}=-\frac{1}{a}\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial t}\,(u\eta)$$
$$+\frac{h_0}{a^2}\,\frac{\partial}{\partial z}\,(1-z^2)\,u\,\frac{\partial u}{\partial z} \qquad \ldots \quad (3)$$

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[1] Airy—"Tides and Waves" Encoy. Metrop. Art., 192, 1845.

[2] McCowan—"On the theory of long waves, etc." Phil. Mag., Series 5, Vol. 35, 250, 1893.

Also Lamb—"Hydrodynamics," Ed. IV, p. 251 and p. 273, (1916).

[3] This problem was attempted in a previous issue of this bulletin (Vol. X, 113. 1918-19) but the solution obtained therein is wrong due to the use of incorrect equations of motion.

where

$$z = \frac{x}{a}$$

Neglecting squares and higher powers of u and η, we have from (2) and (3)

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{gh_0}{a^2} \frac{\partial}{\partial z} (1-z^2) \frac{\partial \eta}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}$$

Assuming that $\eta \propto e^{i\sigma t}$ and putting

$$\sigma^2 = n(n+1) \frac{gh_0}{a^2} \qquad \dots \quad (4)$$

we have

$$\frac{\partial}{\partial z} (1-z^2) \frac{\partial \eta}{\partial z} + n(n+1) \eta = 0$$

whence

$$\eta = C P_n(z) e^{i\sigma t}$$

where n is to be integral since η must be finite when $x = \pm a$, *i.e.* $z = \pm 1$;

$$\therefore \quad \text{From (2), } u = \frac{igC}{a\sigma} P'_n(z) e^{i\sigma t}$$

correct to above order of approximation.

Now, substituting the above values of u and η in the 2nd order terms in (3) and neglecting third and higher powers of u, η and their differential co-efficients, we have after a little simplification.

$$\frac{\partial}{\partial z} (1-z^2) - \frac{n(n+1)}{\sigma^2} \frac{\partial^2 \eta}{\partial t^2}$$

$$= \frac{gC^2}{a^2\sigma^2} \frac{\partial}{\partial z} [2z P'^2_n(z) - 3n(n+1) P_n(z) P'_n(z)] e^{2i\sigma t} \quad \dots \quad (5)$$

Assuming

$$\eta = C P_n(z) e^{i\sigma t} + \Sigma k_r P_r(z) e^{2i\sigma t} \qquad \dots \quad (6)$$

where k_r is so small that its second and higher powers can be neglected, we have from (5) and (6)

$$\sum k_r \left[4n(n+1) - r(r+1) \right] P_r(z)$$

$$= \frac{gC^2}{a^2\sigma^2} \frac{\partial}{\partial z} \left[2z\, P'^{2}_n(z) - 3n(n+1)\, P_n(z)\, P'_n(z) \right] \quad \dots \quad (7)$$

Now, it can be easily proved that

$$P'_n(z) = (2n-1)\, P_{n-1}(z) + (2n-5)\, P_{n-3}(z)$$

$$+ (2n-9)\, P_{n-5}(z) + \dots \quad (8)^{1}$$

$$z\, P'_n(z) = n\, P_n(z) + (2n-3)\, P_{n-2}(z)$$

$$+ (2n-7)\, P_{n-4}(z) + \dots \quad (9)^{2}$$

Also $$P_m(z)\, P_n(z) = \sum_{r=0}^{m} \frac{A_{m-r}\, A_r\, A_{n-r}}{A_{m+n-r}}$$

$$\cdot \frac{2(n+m)+1-4r}{2(n+m)+1-2r}\, P_{n+m-2r}(z) \quad \dots \quad (10)^{3}$$

where

$$n \nless m \quad \text{and} \quad A_m = \frac{\underline{|2m}}{2^m\, \underline{|m}\, \underline{|m}} \quad \dots \quad (11)$$

$$\therefore \quad 2z\, P'^{2}_n(z) - 3n(n+1)\, P_n(z)\, P'_n(z)$$

$$= P'_n(z) \left[2z\, P'_n(z) - 3n(n+1)\, P_n(z) \right]$$

$$= \left[(2n-1)\, P_{n-1}(z) + (2n-5)\, P_{n-3}(z) + (2n-9)\, P_{n-5}(z) + \dots \right]$$

$$\left[-n(3n+1)\, P_n(z) + 2(2n-3)\, P_{n-2}(z) + 2(2n-7)\, P_{n-4}(z) + \dots \right]$$

$$= \left[B_{2n-1}\, P_{2n-1}(z) + B_{2n-3} P_{2n-3}(z) + \dots \right] \quad \dots \quad (12)$$

expressing the product of two Legendre's co-efficients in terms of Legendre's co-efficients by (10).

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[1] Whittaker—Mod. Analysis, p. 303, Result IV.

[2] Whittaker. ibid, p. 324, Ex. 4.

[3] Adams—Proc. Roy. Soc., Vol. 27. Also Whittaker, ibid, p. 325.

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Now, it may be easily seen that

$$B_{2n-1}=-n(2n-1)(3n+1)\ \frac{A_n\ A_{n-1}}{A_{2n-1}}$$

$$B_{2n-3}=-n\ (2n-1)\ (3n+1)\cdot\frac{4n-5}{4n-3}\cdot\frac{A_{n-1}\ A_1\ A_{n-2}}{A_{2n-2}}$$

$$+2\ (2n-1)\ (2n-3)\ \frac{A_{n-1}\ A_{n-2}}{A_{2n-3}}-n\ (2n-5)\ (3n+1)\frac{A_n\ A_{n-3}}{A_{2n-3}}$$

$$B_{2n-5}=-n\ (2n-1)\ (3n+1)\ \frac{4n-9}{4n-5}\ \frac{A_{n-2}\ A_2\ A_{n-3}}{A_{2n-3}}$$

$$+2\ (2n-1)\ (2n-3)\ \frac{4n-9}{4n-7}\ \frac{A_{n-2}\ A_1\ A_{n-3}}{A_{2n-4}}$$

$$-n\ (2n-5)\ (3n+1)\ \frac{4n-9}{4n-7}\ \frac{A_{n-1}\ A_1\ A_{n-4}}{A_{2n-4}}$$

$$+2\ (2n-1)\ (2n-7)\ \frac{A_{n-1}\ A_{n-4}}{A_{2n-5}}$$

$$+2\ (2n-3)\ (2n-5)\ \frac{A_{n-2}\ A_{n-3}}{A_{2n-5}}-n(3n+1)\ (2n-9)\frac{A_n\ A_{n-5}}{A_{2n-5}}$$

etc., etc., where A_m is given by (11).

Again by (8), we have

$$\frac{d}{dz}\ [B_{2n-1}\ P_{2n-1}\ (z)+B_{2n-3}\ P_{2n-3}\ (z)+B_{2n-5}\ P_{2n-5}(z)+\dots]$$

$$=C_{2n-2}\ P_{2n-2}\ (z)+C_{2n-4}\ P_{2n-4}\ (z)+C_{2n-6}\ P_{2n-6}\ (z)+\dots$$

where

$$C_{2n-2}=(4n-3)\ B_{2n-1}$$

$$C_{2n-4}=(4n-7)\ [B_{2n-1}+B_{2n-3}]$$

$$C_{2n-2j}=(4n-4j+1)\ \sum_{r=1}^{j}\ B_{2n-2r+1} \qquad \dots \quad (13)$$

∴ From (7), we have

$$\Sigma k_r \left[4n\,(n+1) - r\,(r+1) \right] P_r\,(z) = \frac{gC^2}{a^2\sigma^2} \sum_{j=1}^{n} C_{2n-2j}\, P_{2n-2j}\,(z)$$

which gives

$$k_{2r+1} = 0 \;(r=0,\,1,\,2 \text{ etc.})$$

$$K_{2n-2j} = \frac{gC^2}{a^2\sigma^2} \quad \frac{C_{2n-2j}}{4n\,(n+1)-(2n-2j)\,(2n-2j+1)}$$

where

$$j=1,\,2 \;\ldots\; n.$$

Hence from (6),

$$\eta = C\,P_n\,(z)\,e^{i\sigma t} + \frac{gC^2}{a^2\sigma^2} \sum_{j=1}^{n} \frac{C_{2n-2j}}{4n(n+1)-(2n-2j)\,(2n-2j+1)}$$

$$P_{2n-2j}(z)\,e^{2i\sigma t} \qquad \ldots \quad (14)$$

where C_{2n-2j} is given by (13).

4. From (14), it is evident that the 2nd order tides are proportional to C^2 and their frequency is double that of the primary disturbance. If the approximation be continued it can be shown that p^{th} order tides are proportinal to C^p and its frequency is p times that of the primary.

5. The following particular cases of interest may be easily deduced from the above results

(a) If $n=1$, i.e. $\frac{\sigma^2 a^2}{gh_0} = 1{\cdot}2$, $\eta = C\,P_1(z)e^{i\sigma t} - \frac{g\,C^2}{2a^2\sigma^2}\,e^{2i\sigma t}$

(b) If $n=2$, i.e. $\frac{\sigma^2 a^2}{gh_0} = 2{\cdot}3$, $\eta = C\,P_2(z)e^{i\sigma t} - \frac{g\,C^2}{a^2\sigma^2}\left\{\frac{3}{2} + 8P_2(z)\right\}e^{2i\sigma t}$

(c) If $n=3$, i.e. $\frac{\sigma^2 a^2}{gh_0} = 3{\cdot}4$,

$$\eta = C\,P_3(z)e^{i\sigma t} - \frac{g\,C^2}{a^2\sigma^2}e^{2i\sigma t}\left\{3 + \frac{720}{49}P_2(z) + \frac{1125}{49}P_4(z)\right\}$$

(*d*) If $n=4$, *i.e.* $\frac{\sigma^2 a^2}{g h_0}=4{\cdot}5$,

$$\eta=\mathrm{C}\,\mathrm{P}_4(z)e^{i\sigma t}-\frac{g\,\mathrm{C}^2}{a^2\sigma^2}e^{2i\sigma t}\left\{5+\frac{2750}{111}\mathrm{P}_2(z)+\frac{465}{11}\mathrm{P}_4(z)+\frac{31850}{627}\mathrm{P}_6(z)\right\}$$

4

ON THE PRODUCT OF BESSEL FUNCTIONS.

BY

K. BASU.

Mr. Abanibhusan Datta (Bull. Cal. Math. Soc., Vol. XII (3), 1921) found out an expression for the product of two Bessel functions in a series of Bessel functions by *two* distinct methods but it was incomplete in as much as he did not lay much stress upon the *coefficients.* The present paper embodies a *third* method of the same, which seems to be an interesting and straight-forward analysis and attempt has been made so that it is applicable to *any number* of Bessel functions. The second section of my treatment involves a method by means of which one can effect indefinite integration of *any number* of products of Bessel functions.

I

Schönholzer established

$$J_{\mu}(z)J_{\nu}(z)=\sum_{n=0}^{\infty}\frac{(-)^{n}\Gamma(\mu+\nu+2n+1)(\tfrac{1}{2}z)^{\mu+\nu+2n}}{n!\,\Gamma(\mu+n+1)\Gamma(\nu+n+1)\Gamma(\mu+\nu+n+1)},$$

for all values of μ and ν Also Neumann proved

$$(\tfrac{1}{2}z)^{n}=\sum_{r=0}^{\infty}\frac{(n+2r).(n+r-1)!}{r!}\,J_{n+2r}(z),$$

whence

$$(\tfrac{1}{2}z)^{\mu+\nu+2n}=\sum_{r=0}^{\infty}\frac{(\mu+\nu+2n+2r).(\mu+\nu+2n+r-1)!}{r!}\,J_{\mu+\nu+2n+2r}$$

$\therefore J_{\mu}(z)J_{\nu}(z)$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-)^{n}\Gamma(\mu+\nu+2n+1)\Gamma(\mu+\nu+2n+2r+1)\Gamma(\mu+\nu+2n+r)}{n!\,r!\,\Gamma(\mu+n+1)\Gamma(\nu+n+1)\Gamma(\mu+\nu+n+1)\Gamma(\mu+\nu+2n+2r)}$$

$$\cdot J_{\mu+\nu+2n+2r}$$

$$=\Omega_{0}J_{\mu+\nu}+\Omega_{-2}J_{\mu+\nu+2}+\Omega_{-4}J_{\mu+\nu+4}+\ldots\ldots+\Omega_{-2K}J_{\mu+\nu+2K}+\&c. \qquad [A].$$

From [A] we can easily determine the coefficients Ω_0, Ω_{-2}, Ω_{-4}, ...Ω_{-2k}, &c., in terms of series of Gamma functions. In fact Ω_0 is obtained by putting $n=0$, $r=0$; Ω_{-2} by putting $n=1$, $r=0$; and $r=1$, $n=0$ and adding up the results. In general Ω_{-2k} is obtained by summing up the terms for $n=0$, $r=k$; $n=1$, $r=k-1$; ..... ; $n=k$, $r=0$. Thus

$$\Omega_0 = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)},$$

$$\Omega_{-2} = \frac{\Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+2)}\left\{\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} + (-)\frac{\Gamma(\mu+\nu+3)}{\Gamma(\mu+2\,\Gamma(\nu+2)}\right\},$$

&c., &c., &c.

$$\Omega_{-2k} = \frac{\Gamma(\mu+\nu+2k+1)}{\Gamma(\mu+\nu+2k)}\left\{\frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu+k)}{0!k!\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)}\right.$$

$$+(-)\frac{\Gamma(\mu+\nu+3)\Gamma(\mu+\nu+k+1)}{1!(k-1)!\Gamma(\mu+2)\Gamma(\nu+2)\Gamma(\mu+\nu+2)}$$

$$+(-)^2\frac{\Gamma(\mu+\nu+5)\Gamma(\mu+\nu+k+2)}{2!(k-2)!\Gamma(\mu+3\,\Gamma(\nu+3)\Gamma(\mu+\nu+3)} + \&c.,$$

$$\left.+(-)^k\frac{\Gamma(\mu+\nu+2k+1)\Gamma(\mu+\nu+2k)}{k!0!\Gamma(\mu+k+1)\Gamma(\nu+k+1)\Gamma(\mu+\nu+k+1)}\right\},$$

$(k+1)$ terms in all.

From the general series for Ω_{-2k} we can write down any coefficient, that is to say, for Ω_0 we take the first term putting $k=0$, for Ω_{-2} we take the first two terms putting $k=1$, and so on. Hence the coefficients are determinable and we establish

$$J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty}\Omega_{-2r}\,J_{\mu+\nu+2r}(z)$$

Again

$$J_\lambda(z)J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty}\Omega_{-2r}\,J_\lambda(z)\cdot J_{\mu+\nu+2r}(z)$$

$$=\Omega_0 J_\lambda(z)\,J_{\mu+\nu}(z) + \Omega_{-2}J_\lambda(z)J_{\mu+\nu+2}(z) + \dots$$

$$+\Omega_{-2k}J_\lambda(z)J_{\mu+\nu+2k}(z) + \&c.$$

Suppose

$$J_\lambda(z)\,J_{\mu+\nu}(z)=\Omega'_0 J_{\lambda+\mu+\nu}(z)+\Omega'_{-2}J_{\lambda+\mu+\nu+2}(z)+\dots$$
$$+\Omega'_{-2k'}J_{\lambda+\mu+\nu+2k'}(z)+\&c.;$$

$$J_\lambda(z)J_{\mu+\nu+2}(z)=\Omega'_{0,2}J_{\lambda+\mu+\nu+2}(z)+\Omega'_{-2,2}J_{\lambda+\mu+\nu+4}(z)+\dots$$
$$+\Omega'_{-2k',2}J_{\lambda+\mu+\nu+2k'+2}(z)+\&c\ ;$$

$$J_\lambda(z)J_{\mu+\nu+4}(z)=\Omega'_{0,4}J_{\lambda+\mu+\nu+4}(z)+\Omega'_{-2,4}J_{\lambda+\mu+\nu+6}(z)+\dots$$
$$+\Omega'_{-2k',4}J_{\lambda+\mu+\nu+2k'+4}(z)+\&c.\ ;$$

&c., &c., &c.

$$J_\lambda(z)J_{\mu+\nu+2k}(z)=\Omega'_{0,2k}J_{\lambda+\mu+\nu+2k}(z)+\Omega'_{-2,2k}J_{\lambda+\mu+\nu+2k+2}(z)$$
$$+\dots+\Omega'_{-2k',2k}J_{\lambda+\mu+\nu+2k+2k'}(z)+\&c.$$

Therefore on substitution

$$J_\lambda(z)J_\mu(z)J_\nu(z)=\Omega_0\Big[\ \Omega'_0J_{\lambda+\mu+\nu}(z)+\Omega'_{-2}J_{\lambda+\mu+\nu+2}(z)+\dots$$
$$+\Omega'_{-2k'}J_{\lambda+\mu+\nu+2k'}(z)+\dots\ \Big]$$

$$+\Omega_{-2}\Big[\ \Omega'_{0,2}J_{\lambda+\mu+\nu+2}(z)+\Omega'_{-2,2}J_{\lambda+\mu+\nu+4}(z)+\dots$$
$$+\Omega'_{-2k',2}J_{\lambda+\mu+\nu+2k'+2}(z)+\dots\Big]$$

$$+\Omega_{-4}\Big[\ \Omega'_{0,4}J_{\lambda+\mu+\nu+4}(z)+\Omega'_{-2,4}J_{\lambda+\mu+\nu+6}(z)+\dots$$
$$+\Omega'_{-2k',4}J_{\lambda+\mu+\nu+2k'+4}(z)+\dots\ \Big]$$

+ &c., &c., &c.

$$+\Omega_{-2k}\Big[\ \Omega'_{0,2k}J_{\lambda+\mu+\nu+2k}(z)+\Omega'_{-2,2k}J_{\lambda+\mu+\nu+2k+2}(z)+\dots$$
$$+\Omega'_{-2k',2k}J_{\lambda+\mu+\nu+2k'+2k}(z)+\dots\ \Big]$$

+ &c

$$=\Omega''_0J_{\lambda+\mu+\nu}+\Omega''_{-2}J_{\lambda+\mu+\nu+2}(z)+\dots+\Omega''_{-2k'}J_{\lambda+\mu+\nu+2k'}(z)$$

+ &c.

where

$$\Omega''_0=\Omega_0\Omega'_0,$$

$$\Omega''_{-2}=\Omega_0\Omega'_{-2}+\Omega_{-2}\Omega'_{0,2},$$

$$\Omega''_{-4}=\Omega_0\Omega'_{-4}+\Omega_{-2}\Omega'_{-2,2}+\Omega_{-4}\Omega'_{0,4},$$

etc., etc., etc.

$$\Omega''_{-2k}=\Omega_0\Omega'_{-2k}+\Omega_{-2}\Omega'_{-2(k-1),2}+\Omega_{-4}\Omega'_{-2(k-2),4}+\cdots$$
$$+\Omega_{-2k}\Omega'_{0,2k},$$

and so on ; that is the new coefficients are expressible in terms of series of products of known coefficients. Proceeding in a similar manner we obtain for $(n+1)$ factors

$$J_{\lambda_1}(z)J_{\lambda_2}(z)J_{\lambda_3}(z)\dots J_{\lambda_{n+1}}(z)=\Omega_0^{(n)}J_{\lambda_1+\lambda_2+\dots+\lambda_{n+1}}(z)$$
$$+\Omega_{-2}^{(n)}J_{\lambda_1+\lambda_2+\dots+\lambda_{n+1}+2}(z)$$

$+$&c., say, where the coefficients $\Omega_0^{(n)}$, $\Omega_{-2}^{(n)}$, ... are determinable from those of the product of n factors. The general formula is

$$\prod_{n+1} J_\lambda(z)=\sum_{k=0}^{\infty}\Omega^{(n)}_{-2k}J_{\Sigma+2k},$$

where Σ stands for $\lambda_1+\lambda_2+\dots+\lambda_{n+1}$.

II

From definition

$$J_p(z)=\sum_{r=0}^{\infty}(-)^r\frac{z^{p+2r}}{2^{p+2r}\,r!\,\Gamma(p+r+1)}$$

whence

$$\int J_p(z)dz = \sum_{r=0}^{\infty} (-)^r \frac{z^{p+2r+1}}{2^{p+2r}\, r!\,\Gamma(p+r+1)}\; \frac{1}{p+2r+1} \tag{1}$$

$$= a_1 J_{p+1} + a_3 J_{p+3} + a_5 J_{p+5} + \&c., \text{ say,}$$

where a_1, a_3, a_5,...are undetermined coefficients. Substituting the values of J_{p+1}, J_{p+3}, &c. in the above we find

$$\int J_p(z)dz = a_1 \left\{ \sum_{r=0}^{\infty} (-)^r \frac{z^{p+1+2r}}{2^{p+1+2r} r!\,\Gamma(p+r+2)} \right\} +$$

$$a_3 \left\{ \sum_{r=0}^{\infty} (-)^r \frac{z^{p+3+2r}}{2^{p+3+2r} r!\,\Gamma(p+r+4)} \right\} + \&c. \tag{2}$$

Comparing the coefficients of z^{p+1}, z^{p+3}. &c., we obtain after necessary simplifications,

$$a_1 = a_3 = a_5 = \ldots = 2$$

Hence

$$\int J_p(z)dz = 2 \sum_{r=0}^{\infty} J_{p+2r+1}(z). \tag{3}$$

Again

$$\int J_\mu(z) J_\nu(z) dz = \sum_{r=0}^{\infty} \int \Omega_{-2r} J_{\mu+\nu+2r}(z) dz.$$

$$= \Omega_0 \int J_{\mu+\nu}(z)dz + \Omega_{-2} \int J_{\mu+\nu+2}(z)dz + \Omega_{-4} \int J_{\mu+\nu+4}(z)dz + \&c.$$

$$= \Omega_0 . 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+1}(z) + \Omega_{-2} . 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+3}(z)$$

$$+ \Omega_{-4} . 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+5}(z) + \&c.$$

$$= 2\Omega_0 \left[J_{\mu+\nu+1}(z) + J_{\mu+\nu+3}(z) + J_{\mu+\nu+5}(z) + \&c. \ldots \right]$$

$$+ 2\Omega_{-2} \left[J_{\mu+\nu+3}(z) + J_{\mu+\nu+5}(z) + J_{\mu+\nu+7}(z) + \&c. \ldots \right]$$

$$+ 2\Omega_{-4} \left[J_{\mu+\nu+5}(z) + J_{\mu+\nu+7}(z) + J_{\mu+\nu+9}(z) + \&c \right]$$

+ &c.

$$+2\Omega_{-2k}\left[J_{\mu+\nu+2k+1}(z)+J_{\mu+\nu+2k+3}(z)+\&c.\ldots\right]+\&c.$$

$$=2(\Omega)'_{0}.J_{\mu+\nu+1}(z)+2(\Omega)'_{-2}J_{\mu+\nu+3}(z)+\ldots$$
$$+2(\Omega)'_{-2k}J_{\mu+\nu+2k+1}(z)+\&c.$$

$$=2\sum_{k=0}^{\infty}(\Omega)'_{-2k}J_{\mu+\nu+2k+1}(z),$$

where

$$(\Omega)'_{0}=\Omega_{0}.$$

$$(\Omega)'_{-2}=\Omega_{0}+\Omega_{-2},$$

$$(\Omega)'_{-4}=\Omega_{0}+\Omega_{-2}+\Omega_{-4},$$

$$\ldots \quad \ldots \quad \ldots$$

$$(\Omega)'_{-2k}=\Omega_{0}+\Omega_{-2}+\Omega_{-4}+\ldots+\Omega_{-2k}, \ \&c.$$

Proceeding in a similar manner with the three-product integral, we obtain

$$\int J_{\lambda}(z)J_{\mu}(z)J_{\nu}(z)dz=2\sum_{k=0}^{\infty}(\Omega)''_{-2k}J_{\lambda+\mu+\nu+2k+1},$$

where

$$(\Omega)''_{0}=\Omega''_{0},$$

$$(\Omega)''_{-2}=\Omega''_{0}+\Omega''_{-2},$$

$$(\Omega)''_{-4}=\Omega''_{0}+\Omega''_{-2}+\Omega''_{-4},$$

$$\ldots \quad \ldots \quad \ldots$$

$$(\Omega)''_{-2k}=\Omega''_{0}+\Omega''_{-2}+\Omega''_{-4}+\ldots+\Omega''_{-2k}, \ \&c.$$

Hence, in general for $(n+1)$–product-integral, we establish

$$\int J_{\lambda_1}(z)J_{\lambda_2}(z)J_{\lambda_3}(z)\ldots J_{\lambda_{n+1}}^{(z)}dz=2\sum_{k=0}^{\infty}(\Omega)^{(n)}_{-2k}J_{\Sigma+2k+1}(z),$$

where Σ has the usual significance.

5

LONGITUDINAL VIBRATIONS OF A HOLLOW CYLINDER.

BY

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1. The longitudinal vibrations of a thin circular cylinder have been discussed at great length by Lord Rayleigh.[1] A second approximation (retaining terms up to the square of the radius of the cylinder), generally known as Pochhammer's solution, has also been obtained by C. Chree.[2] The frequency equation for a solid cylinder of any radius is given in Love's Elasticity. The object of this paper is to obtain the general frequency equation for a hollow solid bounded by two co-axial circular cylinders.

2. We take the axis of the cylinder as the axis of z and (r, θ, z) the cylindrical coordinates of any point Denoting the displacements by u_r, u_θ, u_z, we may assume,[3] as usual,

$$\left.\begin{aligned} u_r &= Ue^{i(az+pt)} \\ u_\theta &= Ve^{i(az+pt)} \\ u_z &= We^{i(az+pt)} \end{aligned}\right\} \qquad \ldots \quad (1)$$

In the case of longitudinal vibrations, we may put U=0 and take V and W independent of θ. We then have

$$\left.\begin{aligned} \Delta &= \left(\frac{\partial U}{\partial r} + \frac{U}{r} + iaW \right) e^{i(az+pt)} \\ \omega_r &= \omega_z = 0,\ 2\omega = \left(iaU - \frac{\partial W}{\partial r} \right) e^{i(az+pt)} \end{aligned}\right\} \qquad \ldots \quad (2)$$

where ω has been put for ω_θ.

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[1] Theory of Sound, Vol. I, Chap. VII.

[2] Quarterly J. of Math, Vol. 21 (1886).

[3] Love's Elasticity, Art. 201.

The equations of motion in terms of Δ and ω are

$$\left.\begin{aligned} &\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+h^2\Delta=0 \\ &\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}-\frac{\omega}{r^2}+k^2\omega=0 \end{aligned}\right\} \quad \ldots \quad (3)$$

where $h^2=\dfrac{p^2\rho}{\lambda+2\mu}-a^2,\ k^2=\dfrac{p^2\rho}{\mu}-a^2$ ... (4)

The solutions of the equations (3) may be written

$$\left.\begin{aligned} &\Delta=\left\{A'J_0(hr)+B'Y_0(hr)\right\}e^{i(rz+pt)} \\ &\omega=\left\{C'J_1(kr)+D'Y_1(kr)\right\}e^{i(rz+pt)} \end{aligned}\right\} \quad \ldots \quad (5)$$

From (2) and (5), we have

$$\frac{\partial U}{\partial r}+\frac{U}{r}+iaW=A'J_0(hr)+B'Y_0(hr)$$

$$iaU-\frac{\partial W}{\partial r}=C'J_1(kr)+D'Y_1(kr)$$

These are satisfied by

$$\left.\begin{aligned} &U=A\frac{\partial}{\partial r}J_0(hr)+B\frac{\partial}{\partial r}Y_0(hr)+CaJ_1(kr)+DaY_1(kr) \\ &W=Aia\,J_0(hr)+BiaY_0(hr) \\ &\qquad+\frac{iC}{r}\frac{\partial}{\partial r}\left\{rJ_1(kr)\right\}+\frac{iD}{r}\frac{\partial}{\partial r}\left\{rY_1(kr)\right\} \end{aligned}\right\}(6)$$

where

$$A=-\frac{A'}{h^2+a^2},\quad B=-\frac{B'}{h^2+a^2}$$

$$C=\frac{2C'}{(k^2+a^2)i},\quad D=-\frac{2D'}{(k^2+a^2)i}$$

The tractions across any surface $r=r$ are given by

$$\widehat{rr}=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r}$$

$$\propto\lambda[A'J_0(hr)+B'Y_0(hr)]$$

$$+2\mu\frac{\partial}{\partial r}\left[A\frac{\partial}{\partial r}J_0(hr)+CaJ_1(kr)+B\frac{\partial}{\partial r}Y_0(hr)+DaY_1(kr)\right]$$

which, after simplification,

$$=A\left\{\lambda'J_0(hr)+\frac{\mu'}{r}J_1(hr)\right\}+B\left\{\lambda'Y_0(hr)+\frac{\mu'}{r}Y_1(hr)\right\}$$

$$+\frac{Ca}{r}\left\{krJ_0(kr)-J_1(kr)\right\}+\frac{Da}{r}\left\{krY_0(kr)-Y_1(kr)\right\},$$

$$\widehat{r\theta}=0,$$

$$\widehat{rz}=\mu\left\{2\omega+2\frac{\partial u_z}{\partial r}\right\}$$

$$\propto\mu\left\{2C'J_1(kr)+2D'Y_1(kr)+2\frac{\partial W}{\partial r}\right\}$$

which reduces to

$$+i\mu\{A.\,2ahJ_1(hr)+B.\,2ahY_1(hr)+C(k^2-a^2)J_1(kr)+D(k^2-a^2).\,Y_1(kr)\},$$

in which we have put

$$\lambda'=\lambda(h^2+a^2)-2\mu h^2$$

$$\mu'=2\mu h$$

3. The notations for Bessels functions used in this paper are those of Gray and Mathews and in the simplifications involved in the following processes, use will be made of the ordinary recurrence-formulae[1] for the functions $J_n(x)$, $Y_n(x)$, $J_n'(x)$, $Y_n'(x)$ and some others derived from them. Use will also be made of the two theorems [1]

$$(i)\quad J_{n+1}(x)Y_n(x)-J_n(x)Y_{n+1}(x)=\frac{1}{x}$$

$$(ii)\quad J_n(x)Y_n'(x)-Y_n(x)J_n'(x)=\frac{1}{x}$$

---

[1] Gray and Mathews, Bessel's Functions, pp. 13, 14, 16.
Nielsen, Theorie der cylinderfunktionen, p. 24.

CASE I.—BOTH BOUNDARIES FREE.

4. If the boundaries $r=a$ and $r=b$ are both free from tractions, we have the following conditions:—

$$A\left[\lambda' J_0(ha)+\frac{\mu'}{a}J_1(ha)\right]+B\left[\lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ha)\right]$$

$$+\frac{Ca}{a}\left[kaJ_0(ka)-J_1(ka)\right]+\frac{Da}{a}\left[kaY_0(ka)-Y_1(ka)\right]=0$$

$$A\left[\lambda' J_0(hb)+\frac{\mu'}{b}J_1(hb)\right]+B\left[\lambda' Y_0(hb)+\frac{\mu'}{b}Y_1(hb)\right]$$

$$+\frac{Ca}{b}\left[kbJ_0(kb)-J_1(kb)\right]+\frac{Da}{b}\left[kbY_0(kb)-Y_1(kb)\right]=0$$

$$A2ahJ_1(ha)+B.2ahY_1(ha)+C(k^2-a^2)J_1(ka)+D(k^2-a^2)Y_1(ka)=0$$

$$A\cdot 2ahJ_1(hb)+B\cdot 2ahY_1(hb)+C(k^2-a^2)J_1(kb)+D(k^2-a^2)Y_1(kb)=0$$

Eliminating the constants A, B, C, D, we get the frequency equation—

$$\begin{vmatrix}
\lambda' J_0(ha)+\frac{\mu'}{a}J_1(ha), & \lambda' J_0(hb)+\frac{\mu'}{b}J_1(hb), & 2ahJ_1(ha), & 2ahJ_1(hb) \\
\lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ha), & \lambda' Y_0(hb)+\frac{\mu'}{b}Y_1(hb), & 2ahY_1(ha), & 2ahY_1(hb) \\
\frac{a}{a}\left\{kaJ_0(ka)-J_1(ka)\right\}, & \frac{a}{b}\left\{kbJ_0(kb)-J_1(kb)\right\}, & (k^2-a^2)J_1(ka), & (k^2-a^2)J_1(kb) \\
\frac{a}{a}\left\{kaY_0(ka)-Y_1(ka)\right\}, & \frac{a}{a}\left\{kbY_0(kb)-Y_1(kb)\right\}, & (k^2-a^2)Y_1(ha), & (k^2-a^2)Y_1(kb)
\end{vmatrix}=0 \qquad (7)$$

or,

$$\left\{\lambda' J_0(ha)+\frac{\mu'}{a}J_1(ha)\right\}\mathrm{I}-\left\{\lambda' J_0(hb)+\frac{\mu'}{b}J_1(hb)\right\}\mathrm{II}$$

$$+2ahJ_1(ha)\,\mathrm{III}-2ahJ_1(hb)\,\mathrm{IV}=0$$

where

$$\mathrm{I}=\left\{\lambda' Y_0(hb)+\frac{\mu'}{b}Y_1(hb)\right\}(k^2-a^2)^2F_{a_1b_1}+\frac{2a^2h}{b}(k^2-a^2)Y_1(ha)$$

$$+\frac{2a^2h}{b}(k^2-a^2)Y_1(hb)\left\{-kbF_{a_1b_0}+F_{a_1b_1}\right\}$$

$$\mathrm{II}=\left\{\lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ka)\right\}(k^2-a^2)^2F_{a_1b_1}$$

$$-2ahY_1(ha)\frac{a}{a}(k^2-a^2)\left\{kaF_{a_0b_1}-F_{a_1b_1}\right\}-2ahY_1(hb)\frac{a}{a}(k_2-a^2)$$

$$\mathrm{III}=-\left\{\lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ha)\right\}\frac{a}{b}(k^2-a^2)$$

$$-\left\{\lambda' Y_0(hb)+\frac{\mu'}{b}Y_1(hb)\right\}\frac{a}{a}(k^2-a^2)\left\{kaF_{a_0b_1}-F_{a_1b_1}\right\}$$

$$+2ahY_1(hb)\cdot\frac{a^2}{ab}\left\{k^2abF_{a_0b_0}-kbF_{a_1b_0}-kaF_{a_0b_1}+F_{a_1b_1}\right\}$$

$$\mathrm{IV}=\left\{\lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ha)\right\}\frac{a(k^2-a^2)}{b}\left\{-kbF_{a_1b_0}+F_{a_1b_1}\right\}$$

$$+\left\{\lambda' Y_0(hb)+\frac{\mu'}{b}Y_1(ha)\right\}\frac{a(k_2-a^2)}{b}$$

$$+2ahY_1(ha)\frac{a^2}{ab}\left\{k^2abF_{a_0b_0}-kbF_{a_1b_0}-kaF_{a_0b_1}+F_{a_1b_1}\right\}$$

where

$$F_{a_rb_s}=-F_{b_sa_r}=J_r(ka)Y_s(kb)-J_s(kb)Y_r(ka)$$

The frequency-equation may be finally put in the form[1]

$$\left(k^2-a^2 \right)^2 \mathrm{F}_{a_1b_1}\left[\lambda'^2\mathrm{G}_{a_0b_0}+\lambda'\mu'\left\{ \frac{1}{a}\,\mathrm{G}a_0b_1+\frac{1}{b}\,\mathrm{G}_{a_1b_0} \right\} + \frac{\mu'^2}{ab}\,\mathrm{G}_{a_1b_1} \right]$$

$$+\frac{2a^2h}{b}\left(k^2-a^2 \right)\left\{ -kb\mathrm{F}_{a_1b_0}+\mathrm{F}_{a_1b_1} \right\} \left\{ \lambda'\mathrm{G}_{a_0b_1}+\frac{\mu'}{a}\,\mathrm{G}_{a_1b_1} \right\}$$

$$+\frac{2a^2h}{a}(k^2-a^2)\left\{ ka\mathrm{F}_{a_0b_1}-\mathrm{F}_{a_1b_1} \right\}\left\{ \lambda'\mathrm{G}_{b_0a_1}-\frac{\mu'}{b}\,\mathrm{G}_{a_1b_1} \right\}$$

$$+\frac{4a^4h^2}{ab}\left\{ k^2ab\mathrm{F}_{a_0b_0}-kb\mathrm{F}_{a_1b_0}-ka\mathrm{F}_{a_0b_1}+\mathrm{F}_{a_1b_1} \right\}\mathrm{G}_{a_1b_1}$$

$$-\frac{4a^2(k^2-a^2)}{ab}=0 \quad \ldots \quad (8)$$

where

$$\mathrm{G}_{a_rb_s} \equiv -\mathrm{G}_{b_sa_r} \equiv \mathrm{J}_r(ha)\mathrm{Y}_s(hb)-\mathrm{J}_s(hb)\mathrm{Y}_r(ha)$$

CASE II.—ONE BOUNDARY RIGID.

5. If the cylinder be free at the surface $r=a$ and rigidly clamped at $r=b$, the boundary conditions are

$$\mathrm{A}\left[\lambda'\mathrm{J}_0(ha)+\frac{\mu'}{a}\,\mathrm{J}_1(ha) \right]+\mathrm{B}\left[\lambda'\mathrm{Y}_0(ha)+\frac{\mu'}{a}\,\mathrm{Y}_1(ha) \right]$$

$$+\frac{\mathrm{C}a}{a}\left[ka\mathrm{J}_0(ka)-\mathrm{J}_1(ka) \right]+\frac{\mathrm{D}a}{a}\left[ka\mathrm{Y}_0(ka)-\mathrm{Y}_1(ka) \right]=0$$

$$\mathrm{A}2ah\mathrm{J}_1(ha)+\mathrm{B}2ah\mathrm{Y}_1(ha)+\mathrm{C}(k^2-a^2)\mathrm{J}_1(ka)+\mathrm{D}(k^2-a^2)\mathrm{Y}_1(ka)=0$$

$$\mathrm{A}h\mathrm{J}_1(hb)+\mathrm{B}h\mathrm{Y}_1(hb)-\mathrm{C}a\mathrm{J}_1(kb)-\mathrm{D}a\mathrm{Y}_1(kb)=0$$

$$\mathrm{A}a\mathrm{J}_0(hb)+\mathrm{B}a\mathrm{Y}_0(kb)+\mathrm{C}k\mathrm{J}_0(kb)+\mathrm{D}k\mathrm{Y}_0(kb)=0$$

---

[1] Here, as also in other cases, the actual process of simplification being rather long, the intermediate steps have been omitted

Eliminating A, B, C, D, we have the frequency-equation

$$\begin{vmatrix} \lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) , & 2\alpha h J_1(ha) , & h J_1(hb) , & \alpha J_0(hb) \\ \lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) . & 2\alpha h Y_1(ha) , & h Y_1(hb) , & \alpha Y_0(hb) \\ \frac{\alpha}{a} \left\{ ka J_0(ka) - J_1(ka) \right\} , & (k^2 - \alpha^2) J_1(ka) , & -\alpha J_1(kb) , & k J_0(kb) \\ \frac{\alpha}{a} \left\{ ka Y_0(ka) - Y_1(ka) \right\} , & (k^2 - \alpha^2) Y_1(ka) , & -\alpha Y_1(kb) , & k Y_0(kb) \end{vmatrix} = 0$$

After simplification, this equation will reduce to

$$\frac{2\alpha^2\lambda'}{b} - \frac{\alpha^2(k^2-\alpha^2)}{ab} - kh(k^2-\alpha^2) F_{a_1 b_0} \left\{ \lambda' G_{a_0 b_1} + \frac{\mu'}{a} G_{a_1 b_1} \right\}$$

$$-\alpha^2(k^2-\alpha^2) F_{a_1 b_1} \left\{ \lambda' G_{a_0 b_1} + \frac{\mu'}{a} G_{a_1 b_0} \right\} + \frac{2\alpha^2 h^2 k}{a} G_{a_1 b_1}$$

$$\left\{ ka F_{a_1 b_0} - F_{a_1 b_0} \right\}$$

$$+ \frac{2\alpha^4 h}{a} G_{a_1 b_0} \left\{ ka F_{a_0 b_1} - F_{a_1 b_1} \right\} = 0 \qquad \ldots \qquad (9)$$

6. The equations (8) and (9) do not admit of exact solutions. Approximate solutions *by trial* may be obtained for assumed values of the ratio $a : b$, by making use of the tables for the values of $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$. The actual work of calculation will of course be very

complicated. The tables of $J_0(x)$ and $J_1(x)$ are given by Meissel[1] and those of $Y_0(x)$ and $Y_1(x)$ by Airey.[2] This method has been adopted by Mr. Southwell[3] in the numerical calculation of some of the approximate values of the period in the case of the transverse vibrations of an annular disc, where, in addition to the ordinary Bessel and Neumann functions, the corresponding functions with imaginary arguments also appear in the frequency equation.

CASE III.—THICKNESS OF THE SHELL VERY SMALL.

7. When the thickness of the shell is very small, we may write $a+da$ for b, expand the functions containing $(a+da)$ in ascending powers of da, and, to a first approximation, neglect all powers of da beyond the first.

Performing these operations in the equation (7), we obtain the frequency-equation for a thin shell of radius a in the form

$$\left|\begin{array}{cccc} \lambda' J_0(ha)+\frac{\mu'}{a}J_1(ha), & -\lambda' h J_1'(ha)-\frac{\mu'}{a^2}J_2(ha), & 2ahJ_1(ha), & 2ah^2J_1'(ha) \\ \lambda' Y_0(ha)+\frac{\mu'}{a}Y_1(ha), & -\lambda' h Y_1(ha)-\frac{\mu'}{a^2}Y_2(ha), & 2ahY_1(ha), & 2ah^2Y_1'(ha) \\ akJ_1'(ka), & ak^2J_1''(ka), & (k^2-a^2)J_1(ka), & (k^2-a^2)J_1'(ka) \\ akY_1'(ka), & ak^2Y_1''(ka), & (k^2-a^2)Y_1(ka); & (ka^2-a^2)Y_1'(ka) \end{array}\right|=0$$

---

[1] Reproduced in Gray and Mathew's Bessel's Functions

[2] Rep. Brit. Assoc. (1914).

[3] Proc. Roy. Soc., Ser. A, Vol. 101 (1922), p. 133.

which gives

$$\frac{(k^2-a^2)^2\lambda'^2h}{ka}\left\{J_1(ha)Y_0(ha)-J_0(ha)Y_1(ka)\right\}$$

$$+\frac{\lambda'\mu'(k^2-a^2)^2}{ka^3}\left\{J_2(ha)Y_0(ha)-J_0(ha)Y_2(ha)\right\}$$

$$+\frac{\mu'^2(k^2-a^2)^2}{ka^4}\left\{J_2(ha)Y_1(ha)-J_1(ha)Y_2(ha)\right\}$$

$$+2ha^2k^2\lambda'(k^2-a^2)H\left\{J_0(ha)Y_1(ha)-J_1(ha)Y_0(ha)\right\}$$

$$+\frac{2\lambda'a^2h^2(k^2-a^2)}{a^2}\left\{J_0(ha)Y_1'(ha)-J_1'(ha)Y_0(ha)\right\}$$

$$+\frac{2\mu'a^2h^3(k^2-a^2)}{a^3}\left\{J_1(ha)Y_1'(ha)-Y_1(ha)J_1'(ha)\right\}$$

$$+\frac{2\lambda'a^2h^3(k^2-a^2)}{ka}\left\{-J_1(ha)Y_1'(ha)+J_1'(ha)Y_1(ha)\right\}$$

$$+\frac{2\mu'a^2h^2(k^2-a^2)}{a^3}\left\{J_1'(ha)Y_2(ha)-J_2(ha)Y_1'(ha)\right\}$$

$$+\ 4a^4h^3k^3H\left\{J_1(ha)Y_1'(ha)-Y_1(ha)J_1'(ha)\right\}=0$$

where

$$\begin{aligned}H&=J_1'(ka)Y_1''(ka)-J_1''(ka)Y_1'(ka)\\
&=\frac{1}{4}\Big[\left\{J_1(ka)-\frac{1}{ka}J_1(ka)\right\}\left\{-3Y_1(ka)+Y_3(ka)\right\}\\
&\qquad\qquad-\left\{Y_0(ka)-\frac{1}{ka}Y_1(ka)\right\}\left\{-3J_1(ka)+J_3(ka)\right\}\Big]\\
&=\frac{1}{4}\Big[3\left\{J_1(ka)Y_0(ka)-J_0(ka)Y_0(ka)\right\}+\left\{J_0(ka)Y_3(ka)\right.\\
&\qquad\left.-Y_0(ka)J_3(ka)\right\}+\frac{1}{ka}\left\{J_3(ka)Y_1(ka)-J_1(ka)Y_3(ka)\right\}\Big]\\
&=\frac{1}{ka}-\frac{1}{k^3a^3}\end{aligned}$$

This can be further simplified into

$$-\frac{(k^2-a^2)^2\lambda'^2}{ka^2}+\frac{2\lambda'\mu'(k^2-a^2)^2}{kh^2a^3}+\frac{\mu'^2(k^2-a^2)}{hka^4}$$

$$-\frac{2a^3k^3\lambda'(k^2-a^2)}{a}\left(\frac{1}{ka}-\frac{1}{k^3a^3}\right)$$

$$+\frac{2\lambda'a^2}{a^4}+\frac{2\mu'a^2h(k^2-a^2)}{a^3}-\frac{2\lambda'a^2h^2(k^2-a^2)}{a^2}$$

$$+\frac{4a^4h^2k^3}{a}\left(\frac{1}{ka}-\frac{1}{k^3a^3}\right)$$

$$-\frac{2\mu'a^2(k^2-a^2)}{a^5}=0$$

or, multiplying throughout by a^5. we have

$$a^3\left\{4a^4h^2k^2-2\lambda'a^2h^2(k^2-a^2)-2\lambda'ka^2(k^2-a^2)-\frac{1}{k}\lambda'^2(k^2-a^2)^2\right\}$$

$$+a^2\left\{\frac{2}{kh^2}\lambda'\mu'(k^2-a^2)^2+2h\mu^2a^2(k^2-a^2)\right\}$$

$$+a\left\{\frac{1}{hk}\mu'^2(k^2-a^2)+\frac{2}{k}\lambda'a^2(k^2-a^2)+2\lambda'a^2-4a^4h^2\right\}$$

$$+2\mu'a^2(k^2-a^2)=0$$

If the tube is of very small bore, and we may neglect all powers of a beyond the first, the frequency equation is

$$a\left\{\frac{1}{hk}\mu'^2(k^2-a^2)+\frac{2}{k}\lambda'a^2(k^2-a^2)+2\lambda'a^2-4a^4h^2\right\}+2\mu'a^2(k^2-a^2)=0.$$

My thanks are due to Prof. S. N. Basu, at whose suggestion I took up the work.

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6

A GENERAL THEOREM FOR THE REPRESENTATION OF X, WHERE X REPRESENTS THE POLYNOMIAL $\frac{x^p-1}{x-1}$.

By

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(1) It is a well known theorem that if p is an odd prime and if X represents the polynomial $\frac{x^p-1}{x-1}$, there is a remarkable transformation of X, which may be expressed as the identity,

$$4X=Y^2-(-1)^{\frac{p-1}{2}}pZ^2,$$

where Y and Z are polynomials in x with integral co-efficients. This identity is known as Gauss's Identity and much has been written about it by different distinguished mathematicians including Gauss, Legendre, L. J. Rogers, G. B. Mathews and others.

There is another identity

$$27X=f\ (U, V, W)$$

where U, V and W are rational integral polynomials in x. This identity has been given by Mr. Eisentein. Without knowing Mr. Eisentein's result[1] I discovered the same identity in a different form.

The object of this paper is to show a general method with the help of which many other formulae of similar type can be easily discovered. The well known Gauss's identity and the cubic identity are only particular cases of this general theorem. It is believed that many other general formulae of transformation will be obtained later on.

This method can be usefully applied only to those values of q for which the cyclotomic section has been completely solved.

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[1] I did not know that the same problem had been worked out by Mr. Eisenstein; it has just been pointed out to me by a referee of London Mathematical Society.

Throughout this paper I have adhered to the notation of Mr. W. Burnside as given in the Proceedings of the London Mathematical Society, 1915.

(2) Let X_1, X_2, $X_3,\ldots,$ X_q be the period values of cyclotomic q-section, [I mean that $X_1, X_2,\ldots,X_q$ have the same meaning as given by G. B. Mathews in proving the Gauss's identity], and let it be supposed that η_0, η_1, $\eta_2,\ldots\eta_{q-1}$ are the roots of the period equation.

(3) I have taken the following notations from the paper of Mr. W. Burnside :

p is an odd prime

q is an odd prime factor, and $p-1=qt$.

ω is an assigned primitive p^{th} root of unity.

a is an assigned primitive root of the congruence $a^{p-1}\equiv 1$ (mod p).

β is an assigned primitive root of the congruence $\beta^{q-1}\equiv 1$ (mod. q).

Each of the $p-1$ primitive p^{th} roots of unity is included just once in the form

$$\omega^{a^{i+rq}} (i=0, 1,\ldots, q-1\,;\ r=0, 1,\ldots, t-1).$$

Put

$$A_i = \sum_{x=0}^{x=t-1} \omega^{a^{i+xq}} (i=0.\ 1\ldots, q-1).$$

Each A_i consists of the sum of t distinct primitive p^{th} roots of unity, and each primitive p^{th} root occurs just once in one of the $A'_i s$. When ω is replaced by ω^{a^q}, each A_i remains unaltered. When ω is replaced by ω^{a}, $A'_i s$ undergo the cyclical permutation

$$(A\ A_1\ ..A_{q-1})$$

If ω' is any root occurring in A_i, then

$$A_i = \sum_{x=0}^{x=t-1} \omega'^{a^{xq}}.$$

In particular since t is even, if A_i contains ω' it will also contain ω'^{-1}, When i is replaced by β_i, the A'_is undergo the permutation

$$\begin{pmatrix} A_0 & A_1 & A_2 \dots A_{q-1} \\ A_0 & A_\beta & A_{2\beta} \dots A_{(q-1)\beta} \end{pmatrix}$$

where the suffixes are reduced (mod. q). This leaves A unchanged and gives a regular circular permutation of the other A'_is.

If A_i and A_j are two distinct A's and if the product of A_i and A_j is formed without reduction, *i.e.*, without taking account of the relation

$$1+\omega+\omega^2+\dots+\omega^{p-1}=0,$$

it will consist of the product of t^2 primitive p^{th} roots because ω^l occurs in A_i, then ω^{l-1} does not occur in A_j. Moreover since A_iA_j is unaltered when ω is replaced by ω^{a^2}, the product can be arranged as the sum of a number of A's.

Hence

$$A_i A_j = \sum_{k=1}^{k=t} C_{ijk} A_k$$

where C's are zeroes or positive integers, such that

$$\sum_{k=1}^{k=t} C_{ijk} = t.$$

The product

$$A_i^2 = t + \sum_{k=1}^{k=t} C_{ijk} A_k,$$

where again the C's are zeroes or positive integers, and

$$\sum_{k=1}^{k=t} C_{ijk} = t-1$$

(4) In particular, the square, cube, etc., of A's can always be represented as the sum of A's. It follows therefore that it is always possible to represent the square, cube, etc., of $\eta_0, \eta_1, \eta_2, \dots$, and η_{q-1} as the sum of a number of $\eta_0, \eta_1, \eta_2, \dots$ and η_{q-1}. Thus we can form q equations which can be always solved uniquely because they are linear simultaneous equations in

$$\eta_0, \eta_1, \eta_2, \dots \text{ and } \eta_{q-1}$$

Therefore X_1, X_2,..., and X_q can always be expressed in the form

$$U+V\eta+W\eta^2+\ldots+M\eta^{q-1}.$$

where U, V, W,...M are polynomials in x with integral coefficients.

What has just been established shows at once that the following is always a possible operation :

$$\begin{aligned}
X_1&=U+V\eta_0+W\eta_0^2+\cdots+M\eta_0^{q-1}.\\
X_2&=U+V\eta_1+W\eta_1^2+\cdots+M\eta_1^{q-1}.\\
X_3&=U+V\eta_2+W\eta_2^2+\cdots+M\eta_2^{q-1}.\\
&\ldots\qquad\ldots\qquad\ldots\qquad\ldots\\
&\ldots\qquad\ldots\qquad\ldots\qquad\ldots\\
X_q&=U+V\eta_{q-1}+\cdots+M\eta_{q-1}^{q-1}.
\end{aligned}$$

Now it is well known that

$$\begin{aligned}
X&=X_1\,X_2\,X_3\cdots X_q.\\
&=(U+V\eta_0+W\eta_0^2+\cdots+M\eta_0^{q-1})\times\\
&\left(U+V\eta_1+W\eta_1^2+\cdots+M\eta_1^{q-1}\right)\times\cdots\\
&\cdots\times\left(U+V\eta_{q-1}+W\eta_{q-1}^2+\ldots M\eta_{q-1}^{q-1}\right).
\end{aligned}$$

$$\begin{aligned}
&=U^q+V^q\eta_0\eta_1\eta_2\cdots\eta_{q-1}+W^q\eta_0^2\eta_1^2\eta_2^2\cdots\eta_{q-1}^2+\cdots\\
&\quad\cdots+M^q\eta_0^{q-1}\eta_1^{q-1}\eta_2^{q-1}\cdots\eta_{q-1}^{q-1}\qquad\cdots\quad\text{(A)}
\end{aligned}$$

the symmetric functions involved in equation (A) can always be determined by the method given in any standard book on theory of equations.

Calculating the symmetric functions in the equation (A) and substituting them in it we find the required formula. Now this general formula will be applied in two particular cases in order to illustrate the use of this method.

When $q=2$, we get

$$\begin{aligned}
X&=X_1\,X_2=(U+V\eta_0)\,(U+V\eta_1)\\
&=U^2+V^2\eta_0\eta_1+UV(\eta_0+\eta_1).
\end{aligned}$$

Substituting the values of η_0 η_1 and $\eta_0+\eta_1$ from the theory of cyclotomic bi-section we get the well known Gauss's Identity.

In order to prove the theorem when $q=3$ let us put for X_1, X_2 and X_3 their values corresponding to cyclotomic periods. Let us suppose that η_0, η_1 and η_2 are the roots of the period equation

$$\eta^3+\eta^2-\frac{p-1}{3}\eta-\frac{1}{9}\left(pa'+\frac{p-1}{3}\right)=0$$

where p is a prime number of the form $6n+1$. Then X_1 is a polynomial of which the coefficients are symmetric functions of the roots of $X=0$, the sum of which makes up $\eta_0=0$. Similar statement holds good for X_2 and X_3.

Let us suppose for a moment that η_0, η_1 and η_2 are subject to the same conditions which are true for η_0 and η_1 in finding the transformation formula

$$4X=Y^2-(-1)^{\frac{p-1}{2}}pZ^2.$$

Then it is evident that the coefficients of X_1 may all be reduced to the form $a'+b\eta_0$. Similarly the coefficients of X_2 and X_3 can also be represented.

Therefore we have identically

$$X_1=U+V\eta_0,$$

$$X_2=U+V\eta_1,$$

$$X_3=U+V\eta_2,$$

where U and V are polynomials in ω with integral coefficients and η_0, η_1 and η_2 are the roots of the period equation

$$\eta^3+\eta^2-\frac{p-1}{3}\eta-\frac{1}{9}\left(pa'+\frac{p-1}{3}\right)=0$$

$$\therefore X=X_1X_2X_3=(U+V\eta_0)(U+V\eta_1)(U+V\eta_2)$$

$$=U^3+\Sigma\eta_0U^2V+\Sigma\eta_0\eta_1UV^2+\eta_0\eta_1\eta_2V^3$$

$$=(3U-V)^3-pV^2(9U-3a'V-V) \qquad \ldots \quad (1)$$

Let us now suppose that the condition to which the investigation given above is subject, has been removed and let 3 be a factor of $p-1$; because p is a prime of the form $6n+1$ this operation is always possible. Let it be supposed that $t=\frac{p-1}{3}$, ω a primitive pth root of unity and a

a primitive root of the congruence $a^{p-1}=1 \pmod p$. Then each of the $p-1$ primitive roots of unity is involved only once in the form

$$\omega^{a^{i+x3}} \qquad (i=0,\ 1,\ 2\,;\ \ x=0,\ 1,\ldots,\ t-1)$$

Put $A_i = \sum_{x=0}^{x=t-1} \omega^{a^{i+3x}} \qquad (i=0,\ 1,\ 2)$

Then each A_i consists of the sum of t distinct primitive pth. root of unity and each primitive pth. root occurs only once. It is very well known that the product of A's can always be represented as the sum of A's and hence in particular the square of A's. Hence it is always possible to represent η_0^2 as the sum of η_0, η_1 and η_2.

$$\therefore \eta_0^2 = \mathrm{m} + a\eta_0 + b\eta_1 + c\eta_2 \qquad \ldots \quad (B)$$

where m, a, b, and c are integers and some of them may be zeroes.

From the theory of cyclotomic tri-section it is evident that the roots η_0, η_1 and η_2 are connected by the following linear relation:—

$$\eta_0 + \eta_1 + \eta_2 = -1 \qquad \ldots \quad (C)$$

By the help of the equations (B) and (C) it is always possible to represent η_1 and η_2 in terms of η_0^2, η_0 and some integers.

$\therefore X_1$ can be represented as $U + V\eta_0 + W\eta_0^2$ where U, V and W are polynomials in x with integral coefficients.

Similarly

$$X_2 = U + V\eta_1 + W\eta_1^2$$

and

$$X_3 = U + V\eta_2 + W\eta_2^2$$

$$\begin{aligned}
\therefore X &= X_1 X_2 X_3 \\
&= (U + V\eta_0 + W\eta_0^2)(U + V\eta_1 + W\eta_1^2) + (U + V\eta_2 + W\eta_2^2) \\
&= U^3 + U^2V(\eta_0 + \eta_1 + \eta_2) + U^2W(\eta_0^2 + \eta_1^2 + \eta_2^2) \\
&\quad + UV^2(\eta_0\eta_1 + \eta_0\eta_2 + \eta_1\eta_2) + UW^2(\eta_0^2\eta_1^2 + \eta_0^2\eta_2^2 \\
&\qquad\qquad + \eta_1^2\eta_2^2) \\
&\quad + UVW(\eta_1\eta_0^2 + \eta_0\eta_1^2 + \eta_0^2\eta_2 + \eta_1^2\eta_2 + \eta_0\eta_2^2 + \eta_1\eta_2^2) \\
&\quad + V^3\eta_0\eta_1\eta_2 + V^2W(\eta_0^2\eta_1\eta_2 + \eta_0\eta_1^2\eta_2 + \eta_0\eta_1\eta_2^2) \\
&\quad + VW^2(\eta_0\eta_1^2\eta_2^2 + \eta_0^2\eta_1^2\eta_2 + \eta_0^2\eta_1\eta_2^2) + W^3\eta_0^2\eta_1^2\eta_2^2
\end{aligned}$$

Now calculating the symmetric coefficients $\Sigma\eta_0$, $\Sigma\eta_0^2$, $\Sigma\eta_0\eta_1$, etc. and substituting the values in the equation just obtained and simplifying it we get

$$
\begin{aligned}
27X = {} & 27U^3 - 27U^2V + (18p+9)U^2W - (9p-9)UV^2 \\
& -9\left(pa' - \frac{2p-2}{3} \right)UVW + 3\left\{ (p-1)^2 + 2pa' + \frac{2p-2}{3} \right\} UW^2 \\
& \qquad + 3\left(pa' + \frac{p-1}{3} \right)V^3 \\
& -3\left(pa' + \frac{p-1}{3} \right)V^2W - (p-1)\left(pa' + \frac{p-1}{3} \right)VW^2 \\
& \qquad + \frac{1}{3}\left(pa' + \frac{p-1}{3} \right)^2 W^3
\end{aligned}
$$

If in this equation W becomes zero, then this formula reduces to the formula (1) obtained above. It is evident that the value of U can not be equal to zero for any prime and hence we can not obtain any formula by supposing U to be equal to zero. It is also evident that whenever W is not zero, V also can not be equal to zero.

Now to establish this theorem in the case when $q=4$ let us put for X_1, X_2, X_3 and X_4 their values corresponding to cyclotomic periods Let us suppose that η_0, η_1 η_2 and η_3 are the roots of the period equation of cyclotomic quarti-section. Then X_1 is a polynomial of which the coefficients are symmetric functions of the roots of $X=0$, the sum of which makes up $\eta_0=0$. Similar statements hold for X_2, X_3 and X_4.

It is possible, as in the previous case. to represent η_0^2 as the sum of η_0, η_1, η_2 ann η_3

$$\therefore \eta_0^2 = m + a\eta_0 + b\eta_1 + c\eta_2 + d\eta_3 \qquad \ldots \quad (D)$$

and

$$\eta_0^3 = m' + a'\eta_0 + b'\eta_1 + c'\eta_2 + d'\eta_3 \qquad \ldots \quad (E)$$

where m, m', a, a', b, b', c, c' and d. d' are integers, some of which may be zeroes.

From the theory of cyclotomic quarti-section it is evident that the roots η_0, η_1, η_2 and η_3 are connected by the following liner relation :—

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1 \qquad \ldots \quad (F)$$

By the help of the equations (D), (E) and (F) it is always possible to represent η_1 and η_2 in terms of η_0^3, η_0^2, η_0 and some integers.

$\therefore X_1$ can be represented as $U+V\eta_0+W\eta_0^2+Y\eta_0^3$, where U, V, W and Y are polynomials in x with integral coefficients.

Similarly

$$X_2=U+V\eta_1+W\eta_1^2+Y\eta_1^3,$$

$$X_3=U+V\eta_2+W\eta_2^2+Y\eta_2^3,$$

and

$$X_4=U+V\eta_3+W\eta_3^2+Y\eta_3^3.$$

$$\therefore X=X_1X_2X_3X_4$$

$$=(U+V\eta_0+W\eta_0^2+Y\eta_0^3)(U+V\eta_1+W\eta_1^2+Y\eta_1^3).$$

$$(U+V\eta_2+W\eta_2^2+Y\eta_2^3)(U+V\eta_3+W\eta_3^2+Y\eta_3^3)$$

$$=U^4+U^3V\Sigma\eta_0+U^3W\Sigma\eta_0^2+U^3Y\Sigma\eta_0^3+U^2V^2\Sigma\eta_0\eta_1+$$

$$U^2W^2\Sigma\eta_0^2\eta_1^2+U^2Y^2\Sigma\eta_0^3\eta_1^3+U^2VW\Sigma\eta_0\eta_1^2+$$

$$U^2VY\Sigma\eta_0\eta_1^3+$$

$$U^2WY\Sigma\eta_0^2\eta_1^3+UV^3\Sigma\eta_0\eta_1\eta_2+UV^2W\Sigma\eta_0\eta_1\eta_2^2+$$

$$UV^2Y\Sigma\eta_0\eta_1\eta_2^3+$$

$$UVW^2\Sigma\eta_0\eta_1^2\eta_2^2+UVY^2\Sigma\eta_0\eta_1^3\eta_2^3+$$

$$UVWY\Sigma\eta_0\eta_1^2\eta_2^3+UW^3\Sigma\eta_0^2\eta_1^2\eta_2^2+$$

$$UW^2Y\Sigma\eta_0^2\eta_1^2\eta_2^3+UWY^2\Sigma\eta_0^3\eta_1^3\eta_2^2+$$

$$UY^3\Sigma\eta_0^3\eta_1^3\eta_2^3+V^4\eta_0\eta_1\eta_2\eta_3+V^3W\Sigma\eta_0\eta_1\eta_2\eta_3^2+$$

$$V^3Y\Sigma\eta_0\eta_1\eta_2\eta_3^3+V^2W^2\Sigma\eta_0\eta_1\eta_2^2\eta_3^2+$$

$$V^2Y^2\Sigma\eta_0\eta_1\eta_2^3\eta_3^3+V^2WY\Sigma\eta_0\eta_1\eta_2^2\eta_3^3+$$

$$VW^3\Sigma\eta_0\eta_1^2\eta_2^2\eta_3^2+VW^2Y\Sigma\eta_0\eta_1^2\eta_2^2\eta_3^3+$$

$$VWY^2\Sigma\eta_0\eta_1^2\eta_2^3\eta_3^3+$$

$$VY^3\Sigma\eta_0\eta_1^3\eta_2^3\eta_3^3+W^4\eta_0^2\eta_1^2\eta_2^2\eta_3^2+W^3Y$$

$$\Sigma\eta_0^2\eta_1^2\eta_2^2\eta_3^3+$$

$$W^2Y^2\Sigma\eta_0^2\eta_1^2\eta_2^3\eta_3^3+WY^3\Sigma\eta_0^2\eta_1^3\eta_2^3\eta_3^3+$$

$$Y^4\eta_0^3\eta_1^3\eta_2^3\eta_3^3.$$

Now calculating the symmetric coefficients $\Sigma\eta_0$, $\Sigma\eta_0\eta_1$, etc, and substituting the values in the equation just obtained and by simplifying it we get

$$\begin{aligned}
X = {} & U^4 - U^3V + (1-2q)U^3W + (3q-3r-1)U^3Y + \\
& qU^2V^2 + (q^2-2r+2s)U^2W^2 + \\
& (3s+q^3+3r^2-3qr-3qs)U^2Y^2 + \\
& (3r-q)U^2VW + (q-2q^2-r+4s)U^2VY + \\
& (2r-q^2+qr-5s)U^2WY - rUV^3 + \\
& (r^2-2qs)UW^3 + (3qrs-r^3-3s^2)UY^3 + \\
& (4qs+qr-3s-3r^2)UVWY + \\
& (r-4s)UV^2W + (2qr-r+s)UV^2Y + \\
& (3s-qr)UW^2V + (2r^2+qs-q^2r-5sr)UVY^2 + \\
& (2qs+sr-r^2)UW^2Y + \\
& (qr^2-2q^2s-sr+4s^2)UWY^2 + sV^4 - sV^3W + \\
& (s-2qs)V^3Y + qsV^2W^2 + s(q^2-2r+2s)V^2Y^2 + \\
& (3sr-qs)V^2WY - srVW^3 + (sr-4s^2)VW^2Y + \\
& (3s^2-qrs)VWY^2 + (sr^2-2qs^2)VY^3 + s^2W^4 \\
& -s^2W^3Y + s^2qW^2Y^2 - s^2rWY^3 + s^3Y^4
\end{aligned}$$

or

$$256X = f(U,V,W,Y)$$

It should be noted here that the period equation of cyclotomic quarti-section is supposed to be

$$\eta^4 + \eta^3 + q\eta^2 + r\eta + s = o\,;$$

and all the symmetric functions involved in the quartic identity given above have been expressed in the terms of the coefficients of the period equation. The coefficients of the period equation, however, can always be determined by the formulae given by A. Cayley, V. S. Le. Resque, Charlotte Angas Scott, W. Burnside.

Putting $q=5$, 6, 7, etc., we can obtain as many identities as we like but the calculation of symmetric functions involved becomes unmanageable.

I have calculated the values of U, V, and W for the primes 13 and 31 in the cubic identity given above. Similarly the values U, V and W for other primes can also be calculated.

Calculation for the prime 13.

It is well known that

$$\eta_0=\omega+\omega^8+\omega^{12}+\omega^5,$$

$$\eta_1=\omega^6+\omega^9+\omega^7+\omega^4,$$

and
$$\eta_2=\omega^{10}+\omega^2+\omega^3+\omega^{11}.$$

We may take any of them. Let us take the first

Then

$$(x-\omega)(x-\omega^8)(x-\omega^{12})(x-\omega^5)=0$$

$$\therefore x^4-\eta_0x^3+(\eta_1+2)x^2-\eta_0x+1=0$$

$$\eta_0+\eta_1+\eta_2=-1 \quad \ldots \quad (1)$$

$$\eta_0{}^2=\eta_2+2\eta_1+4 \quad \ldots \quad (2)$$

Solving the equations (1) and (2), we obtain

$$\eta_1=\eta_0{}^2+\eta_0-3,$$

and
$$\eta_2=-\eta_0{}^2-2\eta_0+2.$$

$$\therefore x^4-\eta_0x^3+(\eta_0{}^2+\eta_0-1)x^2-\eta_0x+1=0.$$

In this case we do not require the value of η_2.

$$\therefore U=x^4-x^2+1.$$

$$V=-x^3+x^2-x.$$

and
$$W=x^2.$$

If we substitute these values in the cubic identity we find that the identity is satisfied.

As this is an identity we may put $x=1$.

and then

$$U=1,$$

$$V=-1,$$

and
$$W=1.$$

Substituting these values in the identity we get

$$27\times13=27\{1+1+9-4-7+14+1+1-4+1\},$$

CALCULATION FOR THE PRIME 31.

$$\eta_0=\omega+\omega^{27}+\omega^{16}+\omega^{29}+\omega^{8}+\omega^{30}+\omega^{4}+\omega^{15}+\omega^{2}+\omega^{23},$$

$$\eta_1=\omega^{3}+\omega^{19}+\omega^{17}+\omega^{25}+\omega^{24}+\omega^{28}+\omega^{12}+\omega^{14}+\omega^{6}+\omega^{7},$$

and $$\eta_2=\omega^{9}+\omega^{26}+\omega^{20}+\omega^{13}+\omega^{10}+\omega^{22}+\omega^{5}+\omega^{11}+\omega^{18}+\omega^{21}.$$

Then $(x-\omega)(x-\omega^{27})(x-\omega^{16})(x-\omega^{29})(x-\omega^{8})(x-\omega^{30})$

$$\times(x-\omega^{4})(x-\omega^{15})(x-\omega^{2})(x-\omega^{23})=0.$$

$$\therefore x^{10}-\eta_0x^9+(\eta_0+2\eta_1+\eta_2+5)x^8-(5\eta_0+3\eta_1+4\eta_2)x^7$$

$$+(10+5\eta_0+8\eta_1+7\eta_2)x^6-(9\eta_0+7\eta_1+9\eta_2+2)x^5$$

$$+(10+5\eta_0+8\eta_1+7\eta_2)x^4-(5\eta_0+3\eta_1+4\eta_2)x^3$$

$$+(5+\eta_0+2\eta_1+\eta_2)x^2-\eta_0x+1=0.$$

$$\eta_0+\eta_1+\eta_2=-1 \qquad \ldots (1)$$

$$\eta_0^2=10+4\eta_1+3\eta_0+2\eta_2 \qquad \ldots (2)$$

Solving the equations (1) & (2) we obtain

$$\eta_1=\frac{\eta_0^2-\eta_0-8}{2},$$

and

$$\eta_2=\frac{6-\eta_0^2-\eta_0}{2}.$$

Substituting these values we get

$$2x^{10}-2\eta_0x^9+(\eta_0^2-\eta_0)x^8+(\eta_0^2-3\eta_0)x^7+(\eta_0^2-5\eta_0-2)x^6$$

$$+(2\eta_0^2-2\eta_0-2)x^5+(\eta_0^2-5\eta_0-2)x^4+(\eta_0^2-3\eta_0)x^3$$

$$+(\eta_0^2-\eta_0)x^2-2\eta_0x+2=0.$$

$$\therefore U=2x^{10}-2x^6-2x^5-2x^4+2,$$

$$V=-2x^9-x^8-3x^7-5x^6-2x^5-5x^4-3x^3-x^2-2x,$$

and $$W=x^8+x^7+x^6+2x^5+x^4+x^3+x^2.$$

Now putting $x=1$,

$$U=-2,$$

$$V=-24,$$

and $$W=8.$$

Substituting these values in the cubic identity we get

$$27\times 31=27\times\tfrac{1}{8}\{-8+96+672+11520-5376-14848$$

$$-110592-36864+32768+122880\}.$$

Here also we see that the identity is satisfied.

Quartic Identity.

I have calculated the values of U, V, W and Y for the primes 13 and 17, which for other primes also can be calculated in a similar way.

For the prime 13.

The period equation of cyclotomic quarti-section for the prime 13 is

$$\eta^4+\eta^3+2\eta^2-4\eta+3=0.$$

The value of quarti-sectional period equation for each prime under 100 has been given by A. Cayley in the Proceedings of London Mathematical Society. And for other primes they can be calculated by the formula given by Miss Scott in the American Journal of Mathematics, VIII.

The formula is as follows :—

$$\eta^4+\eta^3-\{\tfrac{1}{4}(p-1)+l+m\}\eta^2+\tfrac{1}{8}\{p(l-m)-(l+m)\}\eta$$

$$-\tfrac{1}{4}\{p(l-m)^2-(l+m)^2\}=0.$$

But in the quartic identity given above we have supposed the period equation to be

$$\eta^4+\eta^3+q\eta^2+r\eta+s=0.$$

Hence

$$q=-\{\tfrac{1}{4}(p-1)+l+m\},$$

$$r=\tfrac{1}{8}\{p(l-m)-(l+m)\},$$

and

$$s=-\tfrac{1}{4}\{p(l-m)^2-(l+m)^2\}.$$

It is well known that

$$\eta_0=\omega+\omega^3+\omega^9,$$

$$\eta_1=\omega^2+\omega^6+\omega^5,$$

$$\eta_2=\omega^4+\omega^{12}+\omega^{10},$$

and

$$\eta_3=\omega^8+\omega^{11}+\omega^7,$$

Then $$(x-\omega)(x-\omega^3)(x-\omega^9)=0.$$

$$\therefore x^3-\eta_0 x^2+\eta_2 x-1=0$$

$$\eta_0+\eta_1+\eta_2+\eta_3=-1 \qquad \ldots \quad (1)$$

$$\eta_1+2\eta_2-\eta_0^2=0 \qquad \ldots \quad (2)$$

$$\eta_0+3\eta_1+3\eta_3+6-\eta_0^3=0 \qquad \ldots \quad (3)$$

Solving the equations (1), (2) and (3) we obtain

$$\eta_2=\frac{3-2\eta_0-\eta_0^3}{3}.$$

Here in this case we do not require the values of η_1 and η_3.

$$\therefore x^3-\eta_0 x^2+\eta_2 x-1=0 \text{ becomes}$$

$$3x^3-3\eta_0 x^2+(3-2\eta_0-\eta_0^3)x-3=0$$

$$\therefore \mathrm{U}=3x^3+3x-3,$$

$$\mathrm{V}=-(3x^2+2x),$$

$$\mathrm{W}=0,$$

and $$\mathrm{Y}=-x.$$

If we substitute these values in the quartic identity we find that it is satisfied.

Now if we put $x=1$,

$$\mathrm{U}=3,$$

$$\mathrm{V}=-5,$$

$$\mathrm{W}=0,$$

and $$\mathrm{Y}=-1.$$

Substituting these values in the identity we get

$$13=\tfrac{1}{81}\{81+135+0-459+450+0+639+0+450+0-1500+0$$
$$+105+0+0+675+0-1710+0+0+1875+0-1125$$
$$+0+1350+0+0+0+0+60+0+0+0+0+27\}$$

Calculation for the prime 17.

The period equation of cyclotomic quarti-section for the prime 17 is

$$\eta^4+\eta^3-6\eta^2-\eta+1=0.$$

And $$\eta_0=\omega+\omega^{13}+\omega^{16}+\omega^4,$$

$$\eta_1=\omega^3+\omega^5+\omega^{14}+\omega^{12},$$

$$\eta_2=\omega^9+\omega^{15}+\omega^8+\omega^2,$$

and $$\eta_3=\omega^{10}+\omega^{11}+\omega^7+\omega^6.$$

Then

$$(x-\omega)(x-\omega^{13})(x-\omega^{16})(x-\omega^{4})=0.$$

$$\therefore x^4-\eta_0x^3+(2+\eta_1)x^2-\eta_0x+1=0.$$

$$\eta_0+\eta_1+\eta_2+\eta_3=-1 \qquad \ldots \quad (1)$$

$$\eta_0^2=\eta_2+2\eta_1+4 \qquad \ldots \quad (2)$$

$$\eta_0^3=9\eta_0+\eta_1+3\eta_3+3\eta_2 \qquad \ldots \quad (3)$$

Solving the equations (1), (2) and (3) we obtain

$$\eta_1=\tfrac{1}{2}(6\eta_0-3-\eta_0^3).$$

Here we do not require the value of η_2 and η_3.

$$x^4-\eta_0x^3+(2+\eta_1)x^2-\eta_0x+1=0 \text{ becomes}$$

$$x^4-\eta_0x^3+\tfrac{1}{2}(6\eta_0+1-\eta_0^3)x^2-\eta_0x+1=0$$

or

$$2x^4-2\eta_0x^3+(6\eta_0+1-\eta_0^3)x^2-2\eta_0x+2=0.$$

$$U=2x^4+x^2+2,$$

$$V=-2x^3+6x^2-2x,$$

$$W=0,$$

$$Y=-x^2.$$

If we substitute these values in the quartic identity we find that it is satisfied.

Now if we put $x=1$.

$$U=5,$$

$$V=2,$$

$$W=0,$$

and

$$Y=-1.$$

Substituting these values in the identity we get

$$17=\tfrac{1}{16}\{625-250+0+2000-600+0-5250+0+3650+0+40$$
$$+0-80+0+0-280+0+370+0+0+16+0-104$$
$$+0+160+0+0+0+0-26+0+0+0+0+1\}.$$

The quartic identity may also be looked upon as a general formula in quartic forms, because with its help any number of primes of the form $4n+1$, where n is a positive integer, can be represented in a quartic form as has been shown above.

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7

"ON THE COVARIANT CURVES OF A SINGULAR *n–ic*"

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§ 1

If $f \equiv a_x^n = 0$ represents an algebraic curve of the n^{th} degree, its *Hessian*, *Steinerian* and *Cayleyan* are represented by covariants of the form f and are therefore called *covariant curves*. There may, of course, be other curves which are also represented by covariants of the original quantic;[1] but in view of the simplicity of the relations which these curves bear to the original and the detail in which they have been studied, we shall throughout be dealing with them alone whenever we speak of covariant curves. Many beautiful properties of these curves have been established[2] but the central problem in regard to them is however the determination of their Plucker's numbers. This problem has been completely solved and the Plucker's numbers of the covariant curves tabulated in most of the standard treatises on the subject,[3] for the case where the original curve is non-singular. But when the given curve is not non-singular no attempt seems to have been made to calculate systematically all the characteristics of the three curves nor is this surprising, in view of the fact that in addition to the order of each curve we have to determine directly *two* characteristics of each curve or the deficiency of any one[1] of them and *one* other characteristic for each directly. Only some scattered results seem to have been obtained in this direction.[2] We have attempted this problem here and believe that we have been able to solve it successfully.

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[1] For example Zeuthen has considered the curve enveloped by the tangents of the first polars of y (point of Steinerian) at the double points. See

CLEBSCH AND LINDEMANN : *Lecons sur la Géométrie.*

Fr. Trans. t. 2; Ch. I; IV.

[2] Vide *Clebsch-Lindemann*; Ibid Ch. I; Section IV also SALMON, *Higher Plane Curves*. pp 363-8.

[3] See *Encyclopadie der Mathematischen Wiss.* § III C 4. No. 7. pp 339-42.

Coming back to the case of the non-singular primitive curve, it has been pointed out above that the problem of the determination of the Pluckerian characteristics has been completely solved. In fact there are two distinct methods by means of which this has been achieved. We shall briefly indicate them here.

1°. SALMON'S METHOD: Assume that the *Hessian* has in general no double points.[3] The order of the *Hessian* being known *a priori* this assumption enables us to calculate all its characteristics as well as its deficiency and this in virtue of Riemann's theorem, already referred to, will enable the calculation of the characteristics of the other two curves knowing their order and class.

2°. CLEBSCH'S METHOD: This method is indirect and has for its basis the determination of one characteristic of the *Steinerian* in addition to its class and order, *viz.*, the number of its inflexional tangents. In fact, what is done is to take any point and find out the tact-invariant of its first polar and the *Hessian*; this tact-invariant equated to zero would give the point equation of the *Steinerian*. The degree of this tact-invariant in terms of the coefficients of the first polar would then be given by $3(n-2)(5n-11)$ and observing that the tact-invariant also includes the inflexional tangents of the *Steinerian*, the number of these last is given by

$$3(n-2)(5n-11)-3(n-2)^2=3(n-2)(4n-9).$$

The next step is to calculate the deficiency of the *Steinerian*, and then pass on to the other two curves with the aid of Reimann's theorem.

We here indicate another entirely different method of obtaining these results and this consists in proving directly that the *Cayleyan* has no inflexional tangents. This enables us to calculate all the characteristics of this curve as well as its deficiency. We then pass on to the *Steinerian* and *Hessian* and also calculate their Plucker's numbers.

---

[1] In virtue of Riemann's famous theorem relative to the invariance of the deficiency of a curve in all unideterminative transformations, the three covariant curves have the same deficiency.

See "*Clebsch—Lindemann*": Ibid; t. 3.; Ch. I (I)

[2] See *Koehler*: *Bull Soc Math de France* i (1873), pp. 124-9 for the determination of the class of the Steinerian when the given curve has multiple and higher singularities

See also HILTON: *Plane Algebraic Curves*; p. 106.

[3] As regards this assumption see the remarks by Clebsch: *Ibid* t. 2.

§ 2

We first proceed with the case where the original curve is non-singular.

Let C, C_1, C_2, C_3 denote respectively the original curve, its *Hessian*, *Steinerian* and *Cayleyan*. Confining to Clebsch's notation which we shall use throughout, let

n=order of C
k=class „
d=no of double points of C
t=no. of double tangents of C.
r= „ cusps of C.
w= „ inflexional tangents of C.
p=deficiency or genus of C

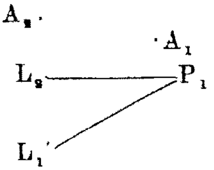

Fig. 1.

and let the same letters with the suoscripts 1, 2 or 3 denote the corresponding characteristics of the *Hessian*, *Steinerian* or *Cayleyan* respectively.

We shall now proceed to prove that $w_2=0$. In fact, if c_3 had an inflexional tangent, then, at the corresponding inflexion-point two of its tangents would be coincident. It requires therefore that corresponding to this inflexional tangent two corresponding points of C_1 and C_2 would coincide or the corresponding points would be double points. It follows therefore that to a double point of C_2 there corresponds a double point of C_1. *If therefore we can show* that double points of C_2 take birth from the fact that to two separate points of C_1, corresponds a single point of C_2 and inversely that to a double point on C_2 correspond two separate points on C_1, we can conclude that to a double point of C_2 cannot correspond a double point of C_1 and hence C_2 has no inflexional tangents.

Nor is it difficult to prove the assumption we have made above. We shall show that the pole of every first polar with two double points must be a node of C_2. Suppose that a first polar curve has two double points A_1 and A_2 (Fig. 1) which must necessarily lie on C_1. Remembering that the polar line of a double point of the first polar touches C_2 at a point of which it is the first polar, we deduce that the polar line of A_1 is a line L_1 touching C_2 at a point P_1. Similarly the polar line of A_2 is a line L_2 touching C_2 at the same point P_1, for, otherwise, one and the same curve will have to be the first polar of two distinct points. Hence the double points A_1 and A_2 give a node on C_2 with L_1 and L_2 as nodal tangents; L_1 and L_2 are necessarily distinct lines for, A_1 and A_2 must be a pair of poles of a line touching C_2 at two points.

Conversely the first polars of any two points, except P_1, on L_1 touch at A_1 and similarly those of any two points on L_2 must touch at another point A_2. Also, the first polar of P_1, considered as a point on L_1, has a double point at A_1 while the same first polar when P_1 is considered a point on L_2 has a double point at A_2. Hence the first polar of a node P_1 on C_2 has two double points A_1 and A_2 lying on C_1. Hence to a double point on C_2 correspond two distinct points on C_1 and *vice versa*; and this is the theorem which we set out to establish.

We can write therefore $w_3=0$

$$\text{but} \quad n_3=3(n-2)(5n-11)$$

$$\text{and} \quad k_3=3(n-1)(n-2).$$

Knowing three of the characteristics of C_3 it is now easy to deduce all the others. We have, in fact,

$$d_3=\frac{9}{2}(n-2)(5n-13)(5n^2-19n+16)$$

$$r_3=18(n-2)(2n-5)$$

$$t_3=\frac{9}{2}(n-2)^2(n^2-2n-1)$$

and we also have

$$p_3=\frac{1}{2}(3n-7)(3n-8)$$

Proceeding next to the curve C_2 we have, in virtue of Riemann's theorem already referred to;

$$p_2=p_3=\frac{1}{2}(3n-7)(3n-8)$$

and *a priori* $n_2=3(n-2)^2$.

$$k_2=3(n-1)(n-2)$$

and these three equations enable us to determine all the other characteristics of C_2. We have actually, with the aid of Plucker's equations:

$$d_2 = \frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$$

$$r_2 = 12(n-2)(n-3)$$

$$t_2 = \frac{3}{2}(n-2)(n-3)(3n^2-3n-8)$$

and $$w_2 = 3(n-2)(4n-9)$$

Similarly for the curve C_1 we have

$$p_1 = p_2 = p_3 = \frac{1}{2}(3n-7)(3n-8)$$

and also $n_1 = 3(n-2)$

so that it only remains to determine one other characteristic of C_1; but we have

$$p_1 = \frac{1}{2}(n_1-1)(n_1-2) - d_1 - r_1$$

and putting $n_1 = 3(n-2)$

$$\frac{1}{2}(n_1-1)(n_1-2) = \frac{1}{2}(3n-7)(3n-8)$$

so that we deduce $d_1 + r_1 = 0$

$$i.e.,\ d_1 = 0 : r_1 = 0$$

It is now easy to determine all the other characteristics of C_1. We have, in fact,

$$k_1 = 3(n-2)(3n-7)$$

$$t_1 = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$$

$$w_1 = 9(n-2)(3n-8)$$

These results may now be tabulated (See Table I).

TABLE I.

—	Hessian.	Steinerian.	Cayleyan.
n	$3(n-2)$	$3(n-2)^2$	$3(n-2)(5n-11)$
k	$3(n-2)(n-7)$	$3(n-1)(n-2)$	$3(n-1)(n-2)$
d	0	$\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$	$\frac{9}{2}(n-2)(5n-3)(5n^2-19n+16)$
t	$\frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$	$\frac{3}{2}(n-2)(n-3)(3n^2-3n-8)$	$\frac{9}{2}(n-2)^2(n^2-2n-1)$
r	0	$12(n-2)(n-3)$	$18(n-2)(2n-5)$
w	$9(n-2)(3n-8)$	$3(n-2)(4n-9)$	0

§ 3

Let us now consider the case where the original curve is not non-singular but has only d double points and r cusps, there being no other higher singularities.

We have seen indirectly in §2 that when the original curve is non-singular the curve C_1 has in general (*i.e.*, if we exclude special relations between the coefficients of the original curve) no double points. A direct proof of this does not as yet seem to have been estiblished.[1] It may, however, happen that a particular relation among the coefficients of the primitive curve may be different from that obtained by expressing the condition for a double point so that C_1 may have a double point without the primitive curve possessing any. Thus, for example, considering the $n-ic$

$$a_x^n \equiv x_3^n + b_x x_3^{n-1} + c_x^2 x_3^{n-2} + d_x^3 x_3^{n-3} + \ldots = 0$$

(b_x, c_x^2, d_x^3 being binary quantics in x_1, x_2) it is easy to show that the first polar of a point P(1,0,0) will have a node at Q(0,0,1) if $b_0 = 0$; $c_0 = c_1 = 0$ and further that the curve does not pass through P. Thus the curve c_1 should pass through Q and it can be shown further that Q is a node on c_1 provided $d_0 = d_1 = 0$. Thus it appears that the *Hessian* of a curve has a double point at (0, 0, 1) without that point lying on the curve at all.[2]

As stated above, this is due to the fact that the particular relation $d_0 = d_1 = 0$ is different from the condition necessary for the possessing of double points.

We shall, however, assume that the *Hessian* has, in general, no double points when the original curve is non-singular.

Now, a double point on the original curve transforms into a double point on C_1 and a cusp into a triple point. Moreover, since the triple point has two distinct branches only and the other touching one of these branches, the triple point is equivalent to *two* nodes and *one* cusp. Thus,

d nodes on C are d nodes on C_1
r cusps „ „ $2r$ nodes and
r cusps on C_1.

---

[1] See "*Clebsch-Lindemann* :" t. 2. Section IV

[2] For another such example see "*Clebsch-Lindemann*" Ibid.

Therefore, when the original curve is not non-singular, we can write

$$d_1 = d + 2r$$

$$r_1 = r$$

Further $$n_1 = 3(n-2)$$

so that we can proceed with the determination of the characteristics of C_1 completely.

We have, in fact,

$$k_1 = 3(n-2)(n-7) - 2d - 7r$$

$$w_1 = 9(n-2)(3n-8) - 6d - 20r$$

$$t_1 = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$$

$$-2d(3n^2 - 27n + 46) - 3r(7n^2 - 63n + 101) + (2d + 7r)^2.$$

We can also calculate the deficiency of C_1 as

$$p_1 = \frac{1}{2}(3n-7)(3n-8) - d - 3r$$

Proceeding next to the curve C_2, we have, in virtue of Riemann's Theorem

$$p_2 = p_1 = \frac{1}{2}(3n-7)(3n-8) - d - 3r$$

Further,

$$n_2 = 3(n-2)^2$$

so that it only remains to determine one other Plucker's number of C_2 when the primitive is not non-singular. But it is a well-known result[1] that the class of the *Steinerian* is in this case equal to

$$k_2 = 3(n-1)(n-2) - 2d - 4r$$

---

[1] See *Hilton* : *ibid*. p 106.

The other Plucker's characteristics now follow easily. We can write

$$d_2=\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)+d+3r$$

$$r_2=12(n-2)(n-3)-2r$$

$$w_2=3(n-2)(4n-9)-6d-14r$$

$$t_2=\frac{3}{2}(n-2)(n-3)(3n^2-3n-8)$$

$$-2d(3n^2-9n+7)-r(12n^2-36n+29)+2(d+2r)^2.$$

It now remains to calculate the Plucker's numbers of the *Cayleyan.* The available data are

$$n_3=3(n-2)(5n-11)$$

$$p_3=p_2=p_1=\frac{1}{2}(3n-7)(3n-8)-d-3r$$

so that, as in the case of C_2, it is sufficient to know one other characteristic of C_3 by direct methods.

A method immediately suggests itself by considering the arguments by means of which we deduced in §2 that $w_3=0$, when the primitive is non-singular. In fact, that proof depended on showing that to a double point on c_2 does not also correspond a double point on C_1 but a pair of distinct points on C_1. Now even when the original curve is not non-singular this argument need not in any way be modified and we can therefore write in this case too

$$w_3=0$$

and the other characteristics of C_3 are now easily determined. We find on actual calculation that

$$d_3=\frac{9}{2}(n-2)(5n-13)(5n^2-19n+16)-2d-6r$$

$$r_3=18(n-2)(2n-5)-3d-9r$$

$$t_3=\frac{9}{2}(n-2)^2(n^2-2n-1)-\delta\phi_1(n)-k\phi_2(n)$$

$$+(A\delta+Bk)^2$$

where ϕ_1, ϕ_2, A and B are obtained by an easy simplification.

Herewith is appended a tabulated list of these Plucker's numbers (See Table II).

TABLE II.

—	Hessian.	Steinerian.	Cayleyan.
n	$3(n-2)$	$3(n-2)^2$	$3(n-2)(5n-11)$
k	$3(n-2)(n-7)-2d-7r$	$3(n-1)(n-2)-2d-4r$	$3(n-1)(n-2)+d+3r$
d	$d+2r$	$\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)+d+3r$	$\frac{9}{2}(n-2\ (5n+3)(5n^2-19n+16)-2d-6r$
t	See the paper proper.		
r	r	$12(n-2)(n-3)-2r$	$18(n-2)(2n-5)-3d-9r$
w	$9(n-2)(3n-8)-6d-20r$	$3(n-2)(4n-9)-6d-14r$	0

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1

EQUITENSE TRANSFORMATIONS ABOUT A FIXED POINT TAKEN AS ORIGIN.

BY

C. E. CULLIS.

[**Summary**. Equitense transformations in ordinary 3-way space Ω_4 (which include *reflexions*, *rotations* and *translations*) are first defined, and are divided into *rigid transformations* and *pseudo-rigid transformations*, —a rigid transformation being an equitense transformation which can be regarded as a resultant of infinitesimal equitense transformations. Those which take place about a fixed finite origin O (divided into *rotations* and *pseudo-rotations* about O) are then discussed in greater detail. In connection with the complete interpretations of rotations and pseudo-rotations about O whose equations are known, special attention may be directed to the theorems of Arts. 4 and 5, in which those interpretations are given in forms characterised by perfect symmetry and freedom from ambiguity. The paper concludes by explaining how pseudo-rigid transformations in Ω_4 can be regarded as rigid transformations in Ω_5,—a reflexion about a plane in Ω_4 being equivalent to a rotation about that plane through two right angles in Ω_5.]

1. Equitense transformations; rigid and pseudo-rigid transformations.

We take O to be a fixed origin accessible to an observer situated in ordinary 3-way space $\Omega=\Omega_4$ (of rank 4); and (OX, OY, OZ) to be a *right-handed* set of rectangular axes drawn from O in Ω. All points and all transformations will be supposed to be *real*; and it will be left to the reader to gather when these restrictions are unnecessary.

We may regard (OX, OY, OZ) as a mathematical abstraction derived from a man standing upright with outstretched arms and looking forwards, O being the base of the head, OX being the right arm, OY being drawn horizontally in the direction of vision, and OZ being drawn vertically upwards through the head. The rotation about OZ which carries OX to OY through a positive right angle is right-handed, and OZ is the right-handed axis of that rotation. If OX′ is the left arm, then (OX′, OY, OZ) are a set of left-handed rectangular axes; the rotation about OZ which carries OX′ to OY through a positive right angle is left-handed; and OZ is the left-handed axis of that rotation. To make the results of this paper applicable when (OX, OY, OZ) are a left-handed set of rectangular axes, the terms 'right-handed' and 'left-handed' must be interchanged whenever they occur.

A projective transformation in Ω is a transformation which converts the points of Ω into the points of an ordinary 3-way space Ω′ coincident with Ω in such a way as to establish a one-one correspondence between:

all finite points of Ω and all finite points af Ω′
all infinite points of Ω and all infinite points of Ω′;

every finite point of Ω being converted into the corresponding finite point of Ω′, and every point in the plane at infinity of Ω being converted into the corresponding point in the plane at infinity of Ω′. If the point P whose co-ordinates with reference to the axes OX, OY, OZ are x, y, z is converted into the point P_1 whose co-ordinates with reference to the same axes are x_1, y_1, z_1, the general equation of such a transformation (as applied to finite points) can be expressed in the form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \text{ (equivalent to } \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M^{-1} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}\text{)}, \qquad \ldots \quad (1)$$

where

$$M = \begin{bmatrix} l_1 & l_2 & l_3 & p \\ m_1 & m_2 & m_3 & q \\ n_1 & n_2 & n_3 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots \quad (2)$$

is an undegenerate square matrix whose elements are all finite.

An *equitense transformation in* Ω is a projective transformation in Ω which leaves the (positive or undirected) distance between every two finite points of Ω unchanged. It necessarily leaves the angles between any two straight lines or any two planes of Ω unchanged. If we put

$$\phi = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \ \phi' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}, \ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots \quad (3)$$

the transformation (1) will be equitense if and only if the equation

$$(x_1'-x_1)^2+(y_1'-y_1)^2+(z_1'-z_1)^2=(x'-x)^2+(y'-y)^2+(z'-z)^2$$

is an identity in x, y, z, x', y', z' when

$$\begin{bmatrix} x_1'-x_1 \\ y_1'-y_1 \\ z_1'-z_1 \end{bmatrix} = \phi . \begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix},$$

i.e. if and only if

$$\phi'\phi = \mathrm{I} = \phi\phi', \qquad \text{or} \qquad \phi' = \phi^{-1} \qquad \dots \quad (4)$$

Accordingly (1) will be the general equation of an equitense transformation (supposed to be real) when and only when ϕ is a real square semi-unit matrix. We then have

$$\det \phi = \pm 1;$$

and the equitense transformation will be called

a *rigid transformation* when $\det \phi = 1$

a *pseudo-rigid transformation* when $\det \phi = -1$.

It can be applied only to those points of Ω which form a configuration S_1. It then converts the points of S into the points of another configuration S_1 lying in Ω.

Clearly all equitense transformations constitute a group G, which is a sub-group of the group of projective transformations; and all rigid transformations constitute a group H, which is a sub-group of G.

The equitense transformation (1) will be called:

(1) the *identical transformation* (which leaves all points of Ω unchanged) when M is the unit matrix of order 4;

(2) an *infinitesimal transformation* (which gives only an infinitesimal displacement to every point of Ω) when we can put

$$\mathrm{M} = \begin{bmatrix} 1+\lambda_1, & l_2, & l_3, & p \\ m_1, & 1+\mu_2, & m_3, & q \\ n_1, & n_2, & 1+\nu_3, & r \\ 0, & 0, & 0, & 1 \end{bmatrix},$$

where all the letters denote infinitesimal scalar numbers, *i.e.* when the difference between M and the unit matrix of order 4 is an infinitesimal matrix;

(3) a *translation* (which can be interpreted to be a rotation through a zero angle about a straight line at infinity) when we can put

$$M=\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that the matrix equation (1) is equivalent to the three scalar equations

$$x_1=x+p,\ y_1=y+q,\ z_1=z+r.$$

In all such cases it is a rigid transformation.

It will be shown (see Art. 4) that every rotation about O is a rotation about a straight line through O, and therefore (see Art 3) a resultant of infinitesimal rotations. Moreover every translation is clearly expressible as a resultant of infinitesimal translations. Hence from *Ex. iv* below we can conclude that:

A rigid transformation is an equitense transformation which is expressible as a resultant of infinitesimal equitense transformations.

Again an equitense transformation will be called one:

about a point A when it leaves the position of A unaltered;

about a straight line (or *axis*) L when it leaves the position of every point of L unaltered;

about a plane p when it leaves the position of every point of *p* unaltered.

All equitense transformations about a given finite point (or straight line or plane) clearly constitute a group.

Ex. i. If an equitense transformation in Ω leaves the positions of two different finite points A and B unaltered, it necessarily leaves the position of every point of the straight line AB unaltered, and is an equitense transformation about AB. If it leaves the positions of three non-collinear finite points A, B, C unaltered, it necessarily leaves unaltered the position of every point in the plane ABC, and is an equitense transformation about that plane If it leaves the positions of three non-coplanar finite points A, B, C, D unaltered, it necessarily leaves unaltered the position of every point of Ω unchanged, and is the identical transformation.

For if it leaves unaltered the points whose co-ordinates are (x, y, z), (x', y', z'), (x'', y'', z''), it also leaves unaltered the point whose co-ordinates are

$$(\lambda x+\mu x'+\nu x'',\ \lambda y+\mu y'+\nu y'',\ \lambda z+\mu z'+\nu z''),\ \text{where } \lambda+\mu+\nu=1.$$

Ex. ii. A given equitense transformation (1) converts O into the point O′ whose co-ordinates with reference to (OX, OY, OZ) are p, q, r; and it converts OX, OY, OZ into the mutually rectangular axes O′X′, O′Y′, O′Z′ whose direction-cosines with reference to (OX, OY, OZ) are (l_1, m_1, n_1), (l_2, m_2, n_2), $(l_3\ m_3\ n_3)$ respectively. The rectangular axes (O′X′, O′Y′, O′Z′) form a right-handed or left-handed set according as as $\det \phi = 1$ or $\det \phi = -1$; and they can clearly be any set of rectangular axes drawn from a finite point in Ω.

Since P_1 in (1) can be any finite point of Ω, and the co-ordinates of P_1 with reference to (O′X′, O′Y′, O′Z′) must be the same as those of P with reference to (OX, OY, OZ), we see by writing

$$(x, y, z) \text{ for } (x_1, y_1, z_1) \text{ and } (x', y', z') \text{ for } (x, y, z)$$

in (1) that if any finite point P of Ω has co-ordinates (x, y, z) with reference to (OX, OY, OZ), then the co-ordinates (x', y', z') of the same point with reference to (O′X′, O′Y′, O′Z′) are given by the equation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \mathrm{M}^{-1} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \text{ (equivalent to } \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathrm{M} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}). \qquad (1')$$

which represents a *transformation of rectangular co-ordinates*. The two sets of rectangular axes (OX, OY, OZ) and (O′X′, O′Y′, O′Z′) are like-handed when and only when $\det \phi = 1$.

Ex. iii. Equitense transformations about a point.

In order that the equitense transformation (1) may be one about the origin O, it is necessary and sufficient that $p = q = r = 0$. The general equations of an equitense transformation about the origin O, about a finite point whose co-ordinates with reference to (OX, OY, OZ) are a, b, c can be expressed in the respective forms

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} x_1 - a \\ y_1 - b \\ z_1 - c \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix},$$

where in each equation the prefactor on the right is a real square semi-unit matrix.

Ex. iv. The matrix M in an equitense transformation (1) can be expressed as a product in the forms

$$\mathrm{M} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \mathrm{P} \\ 0 & 1 & 0 & \mathrm{Q} \\ 0 & 0 & 1 & \mathrm{R} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where
$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

Thus any equitense transformation can be regarded as the resultant of an equitense transformation T about O followed by a translation, and also as the resultant of a translation followed by the same equitense transformation T about O.

2. Equitense bodies or configurations; congruent and pseudo-congruent bodies.

Two finite bodies A and B situated in Ω will be said to be equitense with one another when there exists a one-one correspondence between their points of such a character that the (positive or undirected) distance between any two points of A is equal to the distance between the two corresponding points of B. The following general remarks concerning two such bodies can be established from those properties of equitense transformations which are proved in this paper.

If A and B are *3-dimensional* bodies, and are equitense with one another, there exists in general one and only one equitense transformation T in Ω which converts A into B. According as T is a rigid or pseudo-rigid transformation, A and B will be said to be *congruent* or *pseudo-congruent* with another in Ω (alternatively to be *like* or *unlike* in handedness).

If A and B are *congruent* with one another, we can (in many ways) construct a series of mutually congruent bodies A, C_1, C_2, ... B such that the distance between every pair of corresponding points of any two consecutive bodies of the series is infinitesimal; and each of these bodies can be converted into the next by an infinitesimal equitense transformation. Thus the equitense transformation converting A into B can be regarded as the resultant of a number of successive infinitesimal equitense transformations; or A can be converted into B by a continuous displacement in which it moves as a rigid body.

If A and B are *pseudo-congruent* with one another, the equitense transformation converting A into B cannot be regarded as the resultant of infinitesimal equitense transformations; for such a resultant would necessarily be a rigid transformation. Each of two such bodies may be called an *image* of the other in Ω. If C is any other body in Ω which is equitense with A, then C is congruent with one of the bodies A and B, being convertible into that body by a rigid transformation; and it is pseudo-congruent with the other body, being convertible into that other body by a pseudo-rigid transformation.

In particular cases as when A and B are two spheres or two regular polyhedra, and are equitense with one another, there exist both rigid and pseudo-rigid transformations converting A into B but the correspondences between the points of A and B will be different in different transformations. When the correspondence has been fixed, there is only one transformation as in the general case.

The above remarks (see Art. 7) are applicable only to 3-dimensional bodies in Ω. If A and B are two 2-dimensional (or 1-dimensional) bodies in Ω which are equitense with one another, there exist many equitense transformations in Ω (both rigid and pseudo-rigid) converting A into B, and we can always regard A and B as being congruent to one another in Ω.

3. Equitense transformations about the origin; rotations and pseudo-rotations; perversions.

$$\text{Let } \phi=\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \text{ and } \phi^{-1}=\phi'=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \qquad (1)$$

be respectively a given real square semi-unit matrix of order 3 and its inverse (or conjugate), which is also a real square semi-unit matrix. Also in matrix equations let

$$P=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, P_1=\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, P'=\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, P_1'=\begin{bmatrix} x_1' \\ y_1' \\ z_1' \end{bmatrix}, \quad \ldots \quad (2)$$

Then if (OX, OY, OZ) is a given set of *right-handed* rectangular axes drawn from O, and if these are used as axes of co-ordinates, the *general equation of an equitense transformation about O* in Ω is

$$P_1=\phi \cdot P, \text{ (equivalent to } P=\phi^{-1} \cdot P) \quad \ldots \quad (A)$$

where x, y, z are the co-ordinates of any finite point P with reference to (OX, OY, OZ), and x_1, y_1, z_1 are the co-ordinates with reference to the same axes of the point P_1 in which P is converted by the transformation. If $\det\phi=1$, the equation (A) represents a rigid transformation about O, which will be called a *rotation* about O; if $\det \phi=-1$, it represents a pseudo-rigid transformation about O, which will be called a *pseudo-rotation* about O.

The transformation converts the rectangular axes OX, OY, OZ into the rectangular axes OX′, OY′ OZ′ whose direction cosines with reference to (OX, OY, OZ) are respectively

$$(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3); \qquad \dots \quad (3)$$

and the set of axes (OX′, OY′, OZ′) is right-handed or left-handed according as the transformation is a rotation or a pseudo-rotation. Since OX′, OY′, OZ′ can be any set of rectangular axes drawn from O, and their direction-cosines with reference to (OX, OY, OZ) are known when they are known, we see that there is one and only one equitense transformation about O which converts the axes OX, OY, OZ into another set of rectangular axes drawn from O.

By writing (x, y, z) for (x_1, y_1, z_1) and (x', y', z') for (x, y, z) in (A) we see as in Ex. ii of Art. 1 that the *general equation of a transformation of rectangular co-ordinates* from the axes (OX, OY, OZ) to the axes (OX′, OY′, OZ′) is

$$P' = \phi^{-1}.P, \text{ (equivalent to } P = \phi.P') \qquad \dots \quad (B)$$

where x, y, z are the co-ordinates of any point P with reference to (OX, OY, OZ), and x', y', z' are the co-ordinates of the same point P with reference to (OX′, OY′, OZ′). The two sets of rectangular axes are like-handed when and only when det $\phi = 1$.

Now let OX′, OY′, OZ′ be any second set of rectangular axes drawn from O in Ω, not necessarily those mentioned above; let the equation of a transformation of rectangular co-ordinates from (OX, OY, OZ) to (OX′, OY′, OZ′) be

$$P' = \omega^{-1}.P \text{ (equivalent to } P = \omega.\, P'), \qquad \dots \quad (4)$$

where ω and ω^{-1} are real square semi-unit matrices; and let the points P, P_1 in (A) have co-ordinates;

(x, y, z), (x_1, y_1, z_1) with reference to (OX, OY, OZ);

(x', y', z'), (x_1', y_1', z_1') with reference to (OX′, OY′, OZ′).

Then the equation (4) and the corresponding equation

$$P_1' = \omega^{-1}.P_1 \text{ (equivalent to } P_1 = \omega\, P_1') \qquad \dots \quad (5)$$

show that when OX′, OY′, OZ′ are taken as axes of co-ordinates, the equation of the equitense transformation (A) is

$$P_1' = \omega^{-1}\phi\omega.\, P', \text{ (equivalent to } P' = \omega^{-1}\phi^{-1}\omega.\, P_1'). \qquad \dots \quad (A')$$

It will be more convenient to express this result in another form. If the equation of a given equitense transformation about O is

$$P_1' = \psi . P' \text{ (equivalent to } P' = \psi^{-1} . P), \qquad \dots \quad (C')$$

when OX′, OY′, OZ′ are axes of co-ordinates then the equation of the same equitense transformation is

$$P_1 = \omega\psi\omega^{-1} . P, \text{ (equivalent to } P = \omega\psi\omega^{-1} . P_1), \qquad \dots \quad (C)$$

when OX, OY, OZ are axes of co-ordinates.

The equitense transformation (A) in which

$$\phi = \begin{bmatrix} 1, & 0 & , 0 \\ 0, & \cos\theta, & -\sin\theta \\ 0, & \sin\theta, & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta, & 0, & \sin\theta \\ 0 & , 1, & 0 \\ -\sin\theta, & 0, & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0 & , 1 \end{bmatrix}$$

are *right-handed rotations through the angle θ about the* (*right-handed*) *co-ordinate axes* OX, OY, OZ. Replacing θ by $-\theta$, we obtain the corresponding left-handed rotations through the angle θ about the same axes. If $\theta = \theta_1 + \theta_2 + \theta_3 + \dots$, a right-handed rotation through the angle θ about OZ is the resultant of successive right-handed rotations through the angles θ_1, θ_2, θ_3, ... about OZ. Consequently a rotation about any axis can always be expressed as the resultant of successive infinitesimal rotations about that axis.

The simplest equitense transformations about O are the eight perversions, *viz.* the transformations (A) in which

$$\phi = \begin{bmatrix} \pm 1, & 0, & 0 \\ 0 & , \pm 1. & 0 \\ 0 & , 0 & , \pm 1 \end{bmatrix}$$

They clearly form a complete group. The perversion in which the signs of the successive diagonal elements of ϕ starting from the top are

$$+ \quad + \quad +$$

is the *identical transformation*. The perversions in which those signs are

$$- + + , \quad + - + , \quad + + -$$

are respectively *reflexions about the co-ordinate planes* $x=0$, $y=0$, $z=0$, these being pseudo-rotations. The perversions in which those signs are

$$+\ -\ -\,,\qquad -\ +\ -\,,\qquad -\ -\ +$$

are respectively *reflexions about the co-ordinate axes* OX, OY, OZ, and are also rotations (right-handed or left-handed) through two right angles about those axes. The perversion in which those signs are

$$-\quad -\quad -$$

is a *reflexion about the origin*, which will often be called the *inversion*.

Ex. i. The inversion (or reflexion about O) can be regarded as the resultant of:

(*i*) three successive reflexions about three mutually perpendicular planes (OY, OZ), (OZ, OX), (OX, OY) drawn through O,

(*ii*) a rotation through two right angles about any axis OZ drawn from O followed by (or preceded by) a reflexion about the plane (OX, OY) drawn through O perpendicular to that axis.

Ex. ii. Equation of a reflexion about the plane $lx+my+nz=0$ when l, m, n are direction-cosines.

Let $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, (l, m, n) be the direction-cosines of three rectangular axes OX′, OY′, OZ′ with reference to the given co-ordinate axes OX, OY, OZ; so that in (4) we have

$$\omega=\begin{bmatrix}\lambda_1 & \lambda_2 & l\\ \mu_1 & \mu_2 & m\\ \nu_1 & \nu_2 & n\end{bmatrix}.$$

When OX′, OY′, OZ′ are axes of co-ordinates, the equation of the given plane is $z'=0$, and the equation of a reflexion about it is $z_1'=-z_1$, *i.e.*

$$P_1'=\phi.\,P',\ \text{where}\ \phi=\begin{bmatrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & -1\end{bmatrix}=I-2\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

It follows as in (C′) and (C) that when OX, OY, OZ are axes of co-ordinates, the equation of a reflexion about the given plane $lx+my+nz=0$ is

$$P_1=\psi\,P,\ \text{where}\ \psi=\omega\phi\omega^{-1},$$

$$\text{i.e.}\qquad \psi=I-2\begin{bmatrix}l\\ m\\ n\end{bmatrix}[l,\ m,\ n]=\begin{bmatrix}1-2l^2, & -2lm\,, & -2ln\\ -2ml, & 1-2m^2, & -2mn\\ -2nl, & -2nm\,, & 1-2n^2\end{bmatrix}.$$

This result clearly remains true when the axes OX, OY, OZ are left-handed.

Ex. iii. Equation of a reflexion in the given plane $lx+my+nz+p=0$ when l, m, n are direction-cosines.

If ψ is the same matrix as in Ex. ii, this equation is

$$\begin{bmatrix} x_1+lp \\ y_1+mp \\ z_1+np \end{bmatrix} = \psi . \begin{bmatrix} x+lp \\ y+mp \\ z+np \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2l^2, & -2lm, & -2ln, & -2lp \\ -2ml, & 1-2m^2, & -2mn, & -2mp \\ -2nl, & -2nm, & 1-2n^2, & -2np \\ 0, & 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

Ex. iv. If a plane p can be converted into a plane q by a right-handed rotation through an angle $\frac{1}{2}\alpha$ about any axis, then the resultant of two successive reflexions in p and q is a right-handed rotation through an angle α about that axis.

This can be seen by taking the given axis to be OZ, and p, q to be the planes

$$y=0,\ x \sin \tfrac{1}{2}\alpha - y \cos \tfrac{1}{2}\alpha = 0.$$

Ex. v. Equation of a right-handed rotation through an angle θ about a given axis OZ′ whose direction-cosines are λ, μ, ν.

Let $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, (λ, μ, ν) be the direction-cosines of three right-handed rectangular axes OX′, OY′, OZ′ with reference to the given (right-handed rectangular) co-ordinate axes OX, OY, OZ, so that in (4) we have

$$\omega = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda \\ \mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu \end{bmatrix}, \qquad \text{and } \det \omega = 1.$$

When OX′, OY′, OZ′ are axes of co-ordinates, the equation of the given rotation is

$$P_1' = \Theta . P', \text{ where } \Theta = \begin{bmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix}.$$

It follows as in (C′) and (C) that when OX, OY, OZ are axes of co-ordinates, the equation of the given rotation is

$$P_1 = \phi P, \text{ where } \phi = \omega \Theta \omega^{-1},$$

$$i.e.\qquad \phi=\mathrm{I}+(1-\cos\theta).\begin{bmatrix}\lambda^2-1, & \lambda\mu, & \lambda\nu\\ \mu\lambda, & \mu^2-1, & \mu\nu\\ \nu\lambda, & \nu\mu, & \nu^2-1\end{bmatrix}+\sin\theta\begin{bmatrix}0, & -\nu, & \mu\\ \nu, & 0 & -\lambda\\ -\mu, & \lambda, & 0\end{bmatrix}.$$

This result could also have been deduced from Exs. ii and iv by treating the given rotation as the resultant of two successive reflexions about two planes containing OZ′ which are inclined to one another at an angle $\frac{1}{2}\theta$.

If OX, OY, OZ were a left-handed set of rectangular axes, the same equation would represent a left-handed rotation through an angle θ about OZ′.

Ex. vi. If in Ex. v we express the square semi-unit matrix ϕ in the form

$$\phi=\begin{bmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_1 & n_3\end{bmatrix},$$

we have

$$l_1=1+(1-\cos\theta).(\lambda^2-1)=\lambda^2+\cos\theta.(1-\lambda^2),$$

$$m_2=1+(1-\cos\theta).(\mu^2-1)=\mu^2+\cos\theta.(1-\mu^2),$$

$$n_3=1+(1-\cos\theta).(\nu^2-1)=\nu^2+\cos\theta.(1-\nu^2),$$

$$n_2+m_3=2\,(1-\cos\theta).\,\mu\nu,\quad l_3+n_1=2\,(1-\cos\theta).\,\nu\lambda,\quad m_1+l_2=2\,(1-\cos\theta).\,\lambda\mu.$$

$$n_2-m_3=2\sin\theta.\,\lambda\quad,\quad l_3-n_1=2\sin\theta.\,\mu\quad,\quad m_1-l_2=2\sin\theta.\,\nu.$$

4. The axis and angle of a rotation about the origin O whose equation is given in the general form.

Let the equation of a rotation about the origin O of the right-handed rectangular axes OX, OY, OZ be given in the general form

$$\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}=\begin{bmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad\text{or } \mathrm{P}_1=\phi\cdot\mathrm{P},\quad \dots\quad (\mathrm{D})$$

where ϕ is a given real square semi-unit matrix whose determinant has the value 1; and let

$$\phi(\rho)=\phi-\rho\mathrm{I}=\begin{bmatrix}l_1-\rho, & l_2, & l_3,\\ m_1, & m_2-\rho, & m_3\\ n_1, & n_2, & n_3-\rho\end{bmatrix}\quad \dots\quad (1)$$

be the characteristic matrix of ϕ. The latent roots of ϕ are the roots of the equation $\det\phi(\rho)=0$ or

$$(\rho-1)\{\rho^2-(l_1+m_2+n_3-1)\rho+1\}=0,\quad \dots\quad (2)$$

and because 1 is always a latent root, the square matrix $\phi(1)$ is always degenerate. The position of a finite point P whose coordinates with reference to (OX, OY, OZ) are x, y, z will be unaltered by the rotation if and only if $\phi P = P$, *i.e.*, if and only if

$$\phi(1)\cdot P = 0. \qquad \dots \quad (3)$$

Because $\phi(1)$ is always degenerate, this equation always has at least one finite non-zero solution for the matrix P ; and if P is the corresponding point, the rotation (see Ex. i of Art 1) is one about the straight line OP.

Thus every rotation about O (which may be any finite point) is a rotation about a straight line passing through O.

In the particular case when (D) is the identical transformation, *i.e.*, when $\phi = I$, it leaves the positions of all points of Ω unaltered, and can be interpreted to be a rotation through an angle 0 about any straight line we please passing through O or about any finite straight line whatever.

To obtain all the points whose positions are unaltered by the rotation, or all the solutions of (3), in other cases, we will consider the conjugate reciprocal (or the reciprocal) of $\phi(1)$, which is the symmetric matrix

$$\Phi = \begin{bmatrix} 1+l_1-m_2-n_3, & l_2+m_1 & l_3+n_1 \\ m_1+l_2, & 1+m_2-n_3-l_1, & m_3+n_2 \\ n_1+l_3, & n_2+m_3, & 1+n_3-l_1-m_2 \end{bmatrix} \qquad \dots \quad (4)$$

Because $\qquad \phi(1)\cdot\Phi = \det\phi(1)\cdot I = 0,$

the equation (3) is satisfied when x, y, z are proportional to the 1st, 2nd, 3rd elements in any vertical row of Φ. Again because $\phi(1)$ is degenerate, the rank of Φ cannot exceed 1; therefore by one of the properties of symmetric matrices the radicals

$$Q_1 = \sqrt{1+l_1-m_2-n_3}, \qquad Q_2 = \sqrt{1+m_2-n_3-l_1},$$

$$Q_3 = \sqrt{1+n_3-l_1-m_2} \qquad \dots \quad (5)$$

can be so chosen that

$$\Phi = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} [Q_1, Q_2, Q_3]. \qquad \dots \quad (6)$$

There are then two possible choices of Q_1, Q_2, Q_3, any one of the radicals which is not 0 having whichever we please of its two possible values.

From the equations

$$\begin{array}{lll} m_2n_3-m_3n_2=l_1, & m_3n_1-m_1n_3=l_2, & m_1n_2-m_2n_1=l_3, \\ n_2l_3-n_3l_2=m_1, & n_3l_1-n_1l_3=m_2, & n_1l_2-n_2l_1=m_3 \\ l_2m_3-l_3m_2=n_1, & l_3m_1-l_1m_3=n_2, & l_1m_2-l_2m_1=n_3, \end{array}$$

it follows that

$$\begin{aligned} (n_2+m_3)^2 &= (1+m_2-n_3-l_1)(1+n_3-l_1-m_2), \\ (l_3+n_1)^2 &= (1+n_3-l_1-m_2)(1+l_1-m_2-n_3), \\ (m_1+l_2)^2 &= (1+l_1-m_2-n_3)(1+m_2-n_3-l_1), \qquad \ldots \quad (7) \\ (n_2-m_3)^2 &= (1+l_1+m_2+n_3)(1+l_1-m_2-n_3), \\ (l_3-n_1)^2 &= (1+l_1+m_2+n_3)(1+m_2-n_3-l_1), \\ (m_1-l_2)^2 &= (1+l_1+m_2+n_3)(1+n_3-l_1-m_2), \qquad \ldots \quad (7') \\ (l_3-n_1)(m_1-l_2) &= (1+l_1+m_2+n_3)(n_2+m_3), \\ (m_1-l_2)(n_2-m_3) &= (1+l_1+m_2+n_3)(l_3+n_1), \\ (n_2-m_3)(l_3-n_1) &= (1+l_1+m_2+n_3)(m_1+l_2). \qquad \ldots \quad (7'') \end{aligned}$$

The equations (7) and (7') show that the real factors occurring on the right in them are either all positive or all negative. If they were all negative, it would follow that $3-l_1-m_2-n_3$, the sum of the three diagonal elements of Φ in (4), is negative, but this is impossible because l_1, m_2, n_3 are all real quantities whose numerical values do not exceed 1. We conclude that the real quantities

$$1+l_1-m_2-n_3, \qquad 1+m_2-n_3-l_1, \qquad 1+n_3-l_1-m_2, \qquad \ldots \quad (8)$$

$$1+l_1+m_2+n_3, \qquad 3-l_1-m_2-n_3, \qquad \ldots \quad (8')$$

are all positive. Hence the three radicals Q_1, Q_2, Q_3 and the two radicals

$$Q=\sqrt{1+l_1+m_2+n_3}, \qquad R=\sqrt{3-l_1-m_2-n_3}, \qquad \ldots \quad (5')$$

which satisfy the equations

$$R^2=Q_1^2+Q_2^2+Q_3^2, \quad Q^2+R^2=4, \quad \tfrac{1}{2}(Q^2-R^2)=l_1+m_2+n_3-1, \qquad (9)$$

are all real. We have $R=0$ (or $Q_1=Q_2=Q_3=0$) when and only when $l_1=m_2=n_3=1$, *i.e.*, when and only when (D) is the identical transformation.

If Q_1, Q_2, Q_3 are chosen in accordance with (6), we must have

$$n_2+m_3=Q_2Q_3, \qquad l_3+n_1=Q_3Q_1, \qquad m_1+l_2=Q_1Q_2.$$

The equations (7) are then satisfied, and the equations (7′) and (7″) will also be satisfied if and only if

$$n_2-m_3=\epsilon QQ_1, \qquad l_3-n_1=\epsilon QQ_2, \qquad m_1-l_2=\epsilon QQ_3,$$

where ϵ is either 1 or -1. Whichever choice has been made of Q_1, Q_2, Q_3 (supposing that they are not all equal to 0), we can choose Q so that $\epsilon=1$. Consequently we can, and always will, choose the radicals Q, Q_1, Q_2, Q_3 so that

$$Q_2Q_3=n_2+m_3, \quad Q_3Q_1=l_3+n_1, \quad Q_1Q_2=m_1+l_2, \qquad \dots \quad (10)$$

$$QQ_1=n_2-m_3, \quad QQ_2=l_3-n_1, \quad QQ_3=m_1-l_2\,; \qquad \dots \quad (10')$$

and the equation (6) is then satisfied. Except when (D) is the identical transformation, there are two and only two possible choices of these four radicals, the signs of all being changed when the sign of any one which does not vanish is changed.

That the three quantities (8) and the quantity $3-l_1-m_2-n_3$ are necessarily positive follows immediately from the fact that Φ is a real symmetric matrix whose rank does not exceed 1; for the diagonal elements of such a matrix must all have the same sign.

We now see that when (D) is not the identical transformation, the diagonal elements of Φ do not all vanish; therefore Φ has rank 1, $\Phi(1)$ has rank 2, and the equation (3) has only one distinct non-zero solution given by

$$x:y:z=Q_1:Q_2:Q_3\,; \qquad \dots \quad (11)$$

therefore (D) is a rotation about the uniquely determinate straight line (11), which is the locus of all points whose positions are unaltered. There are two possible axes OA and OA′, drawn from O in opposite directions along that straight line. After choosing Q_1, Q_2, Q_3, Q in accordance with (10) and (10′), and selecting one of the two possible values of R, we will take OA to be the axis

$$\frac{x}{Q_1}=\frac{y}{Q_2}=\frac{z}{Q_3}$$

with direction-cosines

$$\lambda=\frac{Q_1}{R}, \quad \mu=\frac{Q_2}{R}, \quad \nu=\frac{Q_3}{R}. \qquad \dots \quad (12)$$

When (D) is the identical transformation, *i.e.*, when $Q_1=Q_2=Q_3=0$, both $\phi(1)$ and Φ are zero matrices, each having rank 0.

We will next determine the possible angles of rotation.

The three latent roots of ϕ, being the three roots of the equation (3), are

$$1 \text{ and } \tfrac{1}{2}\{(l_1+m_2+n_3-1)\pm\sqrt{(l_1+m_2+n_3-1)^2-4},\}$$

i.e. $$1 \text{ and } \tfrac{1}{2}\{(l_1+m_2+n_3-1)\pm QR\sqrt{-1}\}.$$

Since Q and R are real, *i.e.*, since $\frac{1}{2}(l_1+m_2+n_3-1)$ is not less than -1 and not greater than 1, we can always determine a real angle θ such that

$$\cos\theta=\tfrac{1}{2}(l_1+m_2+n_3-1)=\tfrac{1}{4}(Q^2-R^2)=\tfrac{1}{2}Q^2-1=1-\tfrac{1}{2}R^2\,; \quad \dots \quad (13)$$

and if $i=\sqrt{-1}$, the three latent roots are then

$$1,\ \cos\theta+i\sin\theta,\ \cos\theta-i\sin\theta. \quad \dots \quad (14)$$

This result is in accordance with the facts that every real latent root of a real square semi-unit matrix must be either 1 or -1, and that the latent roots which are not real must occur in pairs of the form $\cos\theta\pm i\sin\theta$, where θ is a real angle. The latent roots (14) are the same for all values of θ satisfying the equation (13). If α is any one angle satisfying the two mutually consistent equations

$$\cos\alpha=\tfrac{1}{2}(l_1+m_2+n_3-1),\ \sin\alpha=\tfrac{1}{2}QR, \quad \dots \quad (15)$$

or any one angle satisfying the two mutually consistent equations

$$2\cos\tfrac{1}{2}\alpha=\epsilon Q,\ 2\sin\tfrac{1}{2}\alpha=\epsilon R, \quad \dots \quad (15')$$

where ϵ is either 1 or -1, then $\theta=\alpha$ and $\theta=-\alpha$ are solutions of (13), and every other solution differs from one of these two by a multiple of 2π.

The three latent roots of ϕ are all real in two particular cases only, *viz.*,

(*i*) when $R=0$, *i.e.* when $l_1+m_2+n_3=3$;

(*ii*) when $Q=0$, *i.e.* when $l_1+m_2+n_3=-1$.

The first particular case occurs when and only when $l_1=m_2=n_3=1$, *i.e.*, when and only when (D) is the identical transformation; the three latent roots being then 1, 1, 1; the angle θ being 0 or a multiple of 2π; and Q_1, Q_2, Q_3 being all 0. The second particular case occurs when and only when the semi-unit matrix ϕ is symmetric but not a unit matrix, as we see from the equations (7′). The angle θ is then an odd multiple of π; the three latent roots of ϕ are 1, -1, -1; and the rotation is not the identical transformation. In all other cases there are two distinct values of θ, if values differing by a multiple of 2π are not considered to be distinct from one another.

By a well known property of real square semi-unit matrices, the equation (D) can be converted by a transformation of rectangular axes into an equation of the form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \ldots \text{ (D}'\text{)}$$

where θ is any solution of the equation (13); and (D′) represents a rotation through an angle θ about the new axis of z, which when $\phi \neq I$ must be one of the two axes OA and OA′. We will verify this without using the general theory of semi-unit matrices, and at the same time determine what angles of rotation are appropriate to each axis.

Supposing that $\phi \neq I$, let OA be the axis (12), and let α be any solution of the two equations (15) or the two equations (15′). Then from Ex. v of Art. 3 we see that the equations of the right-handed rotations $\pm\alpha$ about OA are respectively

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \left\{ \frac{1}{2} \begin{bmatrix} 2l_1, & l_2+m_1 & l_3+n_1 \\ m_1+l_2, & 2m_2, & m_3+n_2 \\ n_1+l_3, & n_2+m_3, & 2n_3 \end{bmatrix} \right.$$

$$\left. \pm \begin{bmatrix} 0, & l_2-m_1, & l_3-n_1 \\ m_1-l_2, & 0, & m_3-n_2 \\ n_1-l_3, & n_2-m_3, & 0 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the equation with the upper sign being the equation (D). Accordingly we have the following theorem in which the axes OX, OY, OZ are right-handed.

Theorem I. *When $\phi = I$, the equation (D) represents the identical transformation. In all other cases it represents a right-handed rotation about an axis OA whose direction-cosines are*

$$\lambda = \frac{Q_1}{R}, \quad \mu = \frac{Q_2}{R}, \quad \nu = \frac{Q_3}{R}$$

and through an angle α which is determined (uniquely except for an arbitrary additive multiple of 2π) by the equations

$$\cos\alpha=\tfrac{1}{2}(l_1+m_2+n_3-1),\ \sin\alpha=\tfrac{1}{2}QR.$$

It is immaterial which sign we ascribe to the radical R, and which of the two possible sets of signs we ascribe to the radicals Q_1, Q_2, Q_3, Q. We can choose Q and R to be both positive, and consider α to be the real angle lying between 0 and π which is uniquely determined by the two equations

$$2\cos\tfrac{1}{2}\alpha=Q,\quad 2\sin\tfrac{1}{2}\alpha=R.$$

5. The pseudo-axis and angle of a pseudo-rotation about the origin O whose equation is given in the general form.

Let the equation of a pseudo-rotation about the origin O of the right-handed rectangular axes OX, OY, OZ be given in the general form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}=\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } P_1=\phi\cdot P, \quad \ldots \quad (E)$$

where ϕ is a given real square semi-unit matrix whose determinant has the value -1. The quantities $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ are the direction-cosines with reference to (OX, OY, OZ) of the left-handed rectangular axes OX′, OY′, OZ′ into which OX, OY, OZ are converted by the pseudo-rotation. Putting $\psi=-\phi$, we can interpret the transformation (E) by comparing it with the transformation

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}=\begin{bmatrix} -l_1, & -l_2, & -l_3 \\ -m_1, & -m_2, & -m_3 \\ -n_1, & -n_2, & -n_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } P_1=\psi\cdot P, \quad \ldots \quad (E')$$

which is a rotation about O. Let (E′) be a right-handed (or left-handed) rotation about an axis OA through an angle θ', and let

$$\theta=\theta'\pm\pi.$$

Then (E) is the resultant of

a right-handed (or left-handed) rotation through an angle θ' about OA;

an inversion, or reflexion about the origin O;

the order in which the two operations are performed being immaterial. Therefore by Ex. *i* of Art 3, it can also be regarded as the resultant of :

a right-handed (or left-handed) rotation through an angle θ about OA ;

a reflexion about the plane through O perpendicular to OA ;

the order in which these two operations are performed being again immaterial. Taking the latter view, we will describe (E) as a right-handed (or left-handed) pseudo-rotation of *angle* θ having OA as a *pseudo-axis* ; a pseudo-axis being a locus of points which suffer a reflexion about the origin.

If $\theta=0$ or is a multiple of 2π, the pseudo-rotation (E) is simply a reflexion about the plane through O perpendicular to OA.

If $\theta=\pm\pi$ or is an odd multiple of π, the pseudo-rotation (E) is the inversion, *i.e.*, a reflexion about the origin O ; and every straight line drawn from O can be regarded as a pseudo-axis. This case occurs when and only when, $\phi=-\mathrm{I}$.

Except when $\phi=-\mathrm{I}$, there are two and only two pseudo-axes OA and OA′, drawn from O in opposite directions.

Applying Art 4 to the rotation (E′), we see that the radicals

$$Q_1=\sqrt{-1+l_1-m_2-n_3},\quad Q_2=\sqrt{-1+m_2-n_3-l_1},$$

$$Q_3=\sqrt{-1+n_3-l_1-m_2}, \qquad \dots \quad (1)$$

$$Q=\sqrt{-1+l_1+m_2+n_3},\quad R=\sqrt{-3-l_1-m_2-n_3} \qquad \dots \quad (1')$$

satisfying the equations

$$R^2=Q_1^2+Q_2^2+Q_3^2,\ Q^2+R^2=-4,\ \tfrac{1}{4}(Q^2-R^2)=l_1+m_2+n_3+1, \qquad (2)$$

are all purely imaginary, and that Q_1, Q_2, Q_3, Q can be so chosen as to satisfy the relations

$$Q_2Q_3=n_2+m_3,\quad Q_3Q_1=l_3+n_1,\quad Q_1Q_2=m_1+l_2, \qquad \dots \quad (3)$$

$$QQ_1=n_2-m_3,\quad QQ_2=l_3-n_1,\quad QQ_3=m_1-l_2. \qquad \dots \quad (3')$$

We have $R=0$ (or $Q_1=Q_2=Q_3=0$) when and only when (E) is the inversion, *i.e.*, $\phi=-\mathrm{I}$; and in all other cases there are two and only two possible choices of the four radicals Q_1, Q_2, Q_3, Q consistent with (3) and (3′).

When the above defined radicals have been chosen in accordance with (3) and (3′) we can deduce from Theorem I the following theorem, in which the axes OX, OY, OZ are right-handed:

Theorem II. *When $\phi=-I$, the equation (E) represents the inversion, i.e., a reflexion about the origin O. In all other cases it represents a right-handed pseudo-rotation about O having a pseudo-axis OA whose direction-cosines are*

$$\lambda=\frac{Q_1}{R}, \quad \mu=\frac{Q_2}{R}, \quad \nu=\frac{Q_3}{R},$$

and angle α which is determined by the equations

$$\cos\alpha=\tfrac{1}{2}(l_1+m_2+n_3+1), \ \sin\alpha=\tfrac{1}{2}QR\ ;$$

i.e., it is the resultant of a right-handed rotation α about OA and a reflexion about the plane through O perpendicular to OA.

Ex. i. When the square semi-unit matrix ϕ in (E) has the values

$$\begin{bmatrix} -1, & 0, & 0 \\ 0, & \cos\theta, & -\sin\theta \\ 0, & \sin\theta, & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta, & 0, & \sin\theta \\ 0, & -1, & 0 \\ -\sin\theta, & 0, & \cos\theta \end{bmatrix},$$

$$\begin{bmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & -1 \end{bmatrix},$$

the equation (E) represents right-handed pseudo-rotations of angle θ about O having OX, OY, OZ respectively as pseudo-axes.

Ex. ii. By a transformation of right-handed rectangular axes it can be deduced from Ex. i that the equation of a right-handed pseudo-rotation of angle θ about O having a pseudo-axis OZ′ with direction-cosines λ, μ, ν is

$$P_1=\phi P,$$

where

$$\phi=-I-(1+\cos\theta)\begin{bmatrix} \lambda^2-1, & \lambda\mu, & \lambda\nu \\ \mu\lambda, & \mu^2-1, & \mu\nu \\ \nu\lambda, & \nu\mu, & \nu^2-1 \end{bmatrix}+\sin\theta\begin{bmatrix} 0, & -\nu, & \mu \\ \nu, & 0, & -\lambda \\ -\mu, & \lambda, & 0 \end{bmatrix}.$$

Ex. iii. If in Ex. ii we put

$$\phi=\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \qquad \delta=det\phi=-1,$$

we have

$$l_1=\delta+(\delta-\cos\theta)\cdot(\lambda^2-1)=\delta\lambda^2+\cos\theta\cdot(1-\lambda^2),$$
$$m_2=\delta+(\delta-\cos\theta)\cdot(\mu^2-1)=\delta\mu^2+\cos\theta\cdot(1-\mu^2),$$
$$n_3=\delta+(\delta-\cos\theta)\cdot(\nu^2-1)=\delta\nu^2+\cos\theta\cdot(1-\nu^2),$$
$$n_2+m_3=2(\delta-\cos\theta)\cdot\mu\nu, \qquad l_3+n_1=2(\delta-\cos\theta)\cdot\nu\lambda, \qquad m_1+l_2=2(\delta-\cos\theta)\cdot\lambda\mu$$
$$n_2-m_3=2\sin\theta\cdot\lambda, \qquad l_3-n_1=2\sin\theta\cdot\mu, \qquad m_1-l_2=2\sin\theta\cdot\nu$$

The corresponding formulae given in Ex. vi of Art 3 are obtained by putting $\delta=1$.

6. Analogies between rotations and pseudo-rotations about the origin.

If we make use of Exs. ii and iii of Art. 5, we can give a direct proof of Theorem II which is strictly analogous to that of Theorem I given in Art. 3, provided that we determine those points which suffer a reflexion about the origin O instead of those points whose positions are unaltered.

If $\delta=\det\phi=\pm 1$, then in both theorems we have

$$Q_1=\sqrt{\delta+l_1-m_2-n_3}, \quad Q_2=\sqrt{\delta+m_2-n_3-l_1}, \quad Q_3=\sqrt{\delta+n_3-l_1-m_2},$$
$$Q=\sqrt{\delta+l_1+m_2+n_3}, \qquad R=\sqrt{3\delta-l_1-m_2-n_3},$$
$$Q_2Q_3=n_2+m_3, \quad Q_3Q_1=l_3+n_1, \quad Q_1Q_2=n_1+l_2,$$
$$QQ_1=n_2-m_3, \quad QQ_2=l_3-n_1, \quad QQ_3=m_1-l_2.$$

In both theorems the latent roots of ϕ are the roots in ρ of the equation

$$(\rho-\delta)\{\rho^2-(l_1+m_2+n_3-\delta)\rho+1\}=0\ ;$$

and both theorems can be proved by determining all the solutions of the matrix equation

$$\phi(\delta)\cdot P=0.$$

In both theorems the matrix $\phi(\delta)$ is degenerate, and its conjugate reciprocal Φ is the symmetric matrix

$$\Phi=\begin{bmatrix} \delta+l_1-m_2-n_3, & l_2+n_1, & l_3+n_1 \\ m_1+l_2, & \delta+m_2-n_3-l_1, & m_3+n_2 \\ n_1+l_3, & n_2+m_3, & \delta+n_3-l_1-m_2 \end{bmatrix}$$

$$=\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}[Q_1,Q_2,Q_3]$$

Ex. i. From Ex ii of Art. 3 it will be seen that in both theorems we can put

$$\phi=\delta\left\{I-2\begin{bmatrix}\lambda_2\\ \mu_2\\ \nu_2\end{bmatrix}[\lambda_2,\mu_2,\nu_2]\right\}\left\{I-2\begin{bmatrix}\lambda_1\\ \mu_1\\ \nu_1\end{bmatrix}[\lambda_1,\mu_1,\nu_1]\right\},$$

where (λ_1,μ_1,ν_1) and (λ_2,μ_2,ν_2) are two sets of direction-cosines. We then have

$$[Q_1,Q_2,Q_3Q]=2\sqrt{\delta}\cdot[\mu_1\nu_2-\mu_2\nu_1,\ \nu_1\lambda_2-\nu_2\lambda_1,\ \lambda_1\mu_2-\lambda_2\mu_1,\ \lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2]$$

$$R=\pm2\sqrt{\delta}\cdot\sqrt{(\mu_1\nu_2-\mu_2\nu_1)^2+(\nu_1\lambda_2-\nu_2\lambda_1)^2+(\lambda_1\mu_2-\lambda_2\mu_1)^2},$$

$$\cos\alpha=\tfrac{1}{2}\delta\cdot(Q^2-R^2)\ ;\ \sin\alpha=\tfrac{1}{2}QR,\ Q^2+R^2=4\delta.$$

Ex. ii. The complete axis and complete pseudo-axis of an equitense transformation about the origin O.

If we define the *complete axis* to be the locus of *all* points whose positions are unaltered by the transformation, and the *complete pseudo-axis* to be the locus of *all* points which suffer a reflexion about O, the general characters of these two loci for any equitense transformation about O are as shown below, where ;

α means an angle which is not a multiple of π ;

L means a straight line drawn through O perpendicular to p

p means a plane drawn through O perpendicular to L.

Rotation of angle θ about O.	Complete axis.	Complete pseudo-axis.
$\theta=\alpha$: (ordinary case.)	a st. line L.	The point O.
$\theta=\pi$: (reflexion about a st. line)	a st. line L.	a plane p.
$\theta=0$: (the identical transformation)	The 3-way space Ω	The point O.

Pseudo-rotation of angle θ about O.	Complete axis.	Complete pseudo-axis.
$\theta=\alpha$: (ordinary case).	The point O.	a st. line L.
$\theta=0$: (reflexion about a plane).	a plane p.	a st. line L.
$\theta=\pi$: (reflexion about O).	The point O.	The 3-way space Ω.

According as the complete axis is

the point O, a st line through O, a plane through O, the 3-way space Ω, the equitense transformation can be regarded as the resultant of successive reflexions about

3, 2, 1, 0 planes through O.

It is a rotation in the second and fourth cases, and a pseudo-rotation in the first and third cases.

7. Pseudo-rigid transformations in Ω_4 regarded as rigid transformations in Ω_5.

The principles and methods explained in the foregoing articles can be extended to ordinary metrical space Ω_{n+1} of n dimensions, where n is any positive integer. Any particular set of n rectangular axes drawn from a finite point O in Ω_{n+1} can be regarded as right-handed ; and by reversing the direction of one of the axes we obtain a second set of rectangular axes in Ω_{n+1} which is left-handed. The choice of a standard set of right-handed rectangular axes in Ω_{n+1} will depend on some standard n-dimensional configuration in Ω_{n+1}.

Let Ω_4 be the ordinary 3-way space Ω which has hitherto been considered, (OX, OY, OZ) being a set of three rectangular axes in Ω_4 ; and let (OX, OY, OZ, OW) be a set of four rectangular axes in an ordinary 4-way metrical space Ω_5 which contains Ω_4. Then if the first of the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & a \\ m_1 & m_2 & m_3 & b \\ n_1 & n_2 & n_3 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 & 0 \\ m_1 & m_2 & m_3 & 0 & 0 \\ n_1 & n_2 & n_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ 1 \end{bmatrix} \quad \ldots \ (1)$$

represents a pseudo-rigid transformation in Ω_4, the second equation represents a rigid transformation in Ω_5. Moreover if S is any 3-dimensional body in Ω_4 which is converted into S_1 by the first transformation,

w being 0 for all points of S, then the second transformation also converts S into S_1. Consequently S can be converted into S_1 by a succession of infinitesimal equitense transformations in Ω_5, *i.e.* by a continuous displacement in which it moves as a rigid body; but in general it will be entirely outside the space Ω_4 in all the intermediate positions.

Again of the first of the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \dots \quad (2)$$

represents a pseudo-rotation in Ω_4 about O, the second equation represents a rotation in Ω_5 about O; and if S is any 3-dimensional body in Ω_4 which is converted into S_1 by the pseudo-rotation in Ω_4, then S will also be converted into S_1 by the rotation in Ω_5 represented by the second equation. The two transformations have the same complete axis, but the complete pseudo-axis of the second transformation is the space determined by the complete pseudo-axis of the first transformation and the new co-ordinate axis OW. The pairs of transformations next considered are representative cases of (2).

The two equitense transformations about O represented by the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \cos\theta, & -\sin\theta, & 0, & 0 \\ \sin\theta, & \cos\theta, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \ldots \quad (3)$$

are equivalent when applied to a 3-dimensional body in Ω_4. The first is a pseudo-rotation of angle θ in Ω_4 having OZ as a pseudo-axis ; the second is a rotation in Ω_5 which is the resultant of a rotation θ about the plane (OZ, OW) and a rotation π about the plane (OX, OY). In the ordinary case when θ is not a multiple of π, the first transformation has the straight line OZ as complete pseudo-axis, the point O as complete axis, whilst the second transformation has

the plane (OZ,OW) as complete pseudo-axis, the point O as complete axis.

In the particular case when $\theta=\pi$ (or is an odd multiple of π), the first of the transformations (3) is a reflexion about O in Ω_4 ; and the second is a reflexion about O in Ω_5, which is also the resultant of a rotation π about the plane (OZ,OW) and a rotation π about the plane (OX, OY). The first transformation has

the 3-way space Ω_4 as complete pseudo-axis, the point O as complete axis ; whilst the second transformation has

the 4-way space Ω_5 as complete pseudo-axis, the point O as complete axis.

In the particular case when $\theta=0$ (or is a multiple of 2π), the two transformations (3) become

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}. \quad \ldots \quad (4)$$

The first transformation is a reflexion in Ω_4 about the plane (OX, OY) ; and the second is a rotation in Ω_5 through two right angles about the the plane (OX, OY). The first transformation has the straight line OZ as complete pseudo-axis, the plane (OX, OY) as complete axis ;

whilst the second transformation has

the plane (OZ, OW) as complete pseudo-axis, the plane (OX, OY) as complete axis.

From (4) it will be seen that, so far as events in Ω_4 are concerned, a reflexion in Ω_4 about a plane p is equivalent to a rotation through two right angles about p in Ω_5. Thus if a man existing in Ω_4 and occupying the configuration H surveys his image H′ formed by reflexion about a plane mirror, he will know that he could be carried from H to H′ by a rotation in Ω_5 about the plane of the mirror through two right angles, *i.e.*, by a continuous rigid movement in Ω_5 ; but in the execution of this movement his Ω_4-existence would cease in all the configurations intermediate between H and H′ ; *i.e.*, in every such intermediate position he would be entirely outside the space Ω_4 to which his existence is confined. Using definitions appropriate to Ω_4, the right arm of H will be converted into the left arm of H′ ; but if we used definitions appropriate to Ω_5, the right arm of H would be converted into the right arm of H′. In the latter case we regard H and H′ as two different aspects of the same 3-dimensional entity, and distinguish between the two sides of that entity in Ω_5, a distinction which is impossible in Ω_4. Similarly in speaking of the right-hand and left-hand edges of a printed page, we use definitions appropriate to 2-way space ; but in order to speak of the right-hand and left-hand edges of a printed leaf, we should have to distinguish between the front page and back page, and use definitions appropriate to 3-way space.

The distinction between congruence and pseudo-congruence (or between a right-handed and left-handed set of three rectangular axes) which occurs in Ω_4, disappears in Ω_5 ; and both the spaces Ω_4 and Ω_5 have definitions of right-handedness and left-handedness which are peculiar to them. Similarly the distinction between congruence and pseudo-congruence which occurs in Ω_{n+1}, disappears in Ω_{n+2} ; and every space Ω_{n+1} has definitions of right-handedness and left-handedness which are peculiar to it.

2

ON THE EVALUATION OF SOME FACTORABLE CONTINUANTS.

Part II.

By

SATISH CHANDRA CHAKRABARTI, M.Sc.

In the first part of this paper, published in the *Bulletin of the Calcutta Mathematical Society, Vol. XIII, Nos. 1* and *2*, pp. 71-84 towards the end of the Article 7, there have been given some operations for evaluating a factorable continuant. In combining these operations we have got some identities which are treated in Arts. 3 and 4 of the present paper. In the paper[1] of Mr. Haripada Datta in which the above continuant occurs, there has been given another factorable continuant which has been evaluated in Art. 1 determinantally. In combining the operations given in Art. 1, we have got some more identities which have been established in Arts. 5, 6 and 7. In Art. 2 we have considered the general case of the identities which occur in Arts. 1, 2 and 3 of the first part.

1. The continuant

$$\begin{vmatrix} 1 & x & & & & & \\ 1+a^{n-1}y, & 1+y & x & & & & \\ & (a-1)(a^{n-1}-1)y, & 1+ay, & x & & & \\ & & (1+y)(1+a^2y)a, & 1+a^2y, & x & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & (a^{n-1}-1)(a-1)ya^{2n-4}, & 1+a^{2n-3}y, & x \\ & & & & & (1+a^{n-2}y)\,a^{n-1}, & 1 \end{vmatrix}_{2n}$$

$$=\{(1+y)(1+ay)(1+a^2y)\ldots\ldots(1+a^{2n-3}y)\}$$
$$\times\{(1-x)\,(1-ax)\,(1-a^2x)\ldots\ldots(1-a^{n-1}x)\}$$

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[1] Haripada Datta, "On the Failure of Heilermann's Theorem," *Proc. Edin. Math, Soc.* Vol. 35, part 2, 1916-17 or Univ. Edin. Math. Depart. Session 1917, Research paper No. 7, pp. 10.

Here the elements, except the first and the last, of the lower minor diagonal, are given by $e_{2m}=(1+a^{m-2}y)(1+a^{m+n-2}y)a^{m-1}$ and $e_{2m+1}=(a^m-1)(a^{n-m}-1)ya^{2m-2}$, where e_r denotes the element of this diagonal in the rth row.

Let us first consider the particular case when $n=3$, *viz.*

$$\begin{vmatrix} 1 & x & & & & \\ 1+a^2y & 1+y, & x & & & \\ & (a-1)(a^2-1)y, & 1+ay, & x & & \\ & & (1+y)(1+a^3y)a, & 1+a^2y, & x & \\ & & & (a^2-1)(a-1)ya^2, & 1+a^3y, & x \\ & & & & (1+ay)a^2, & 1 \end{vmatrix}$$

On this perform the first operation

$$-a^5(a^2-1)(a-1)y^3(1+y)(1+ay)\mathrm{col}_6+a^3(a^2-1)(a-1)y^2(1+y)\mathrm{col}_5$$
$$+a(a^2-1)y(1+y)\mathrm{col}_4-(a^2-1)y\mathrm{col}_3-\mathrm{col}_2+\mathrm{col}_1$$

This enables us to remove $1-x$ from the last column and then subtracting the first column from the last we can remove another factor $-(1+y)$ from that column and write the co-factor in the form

$$\frac{1}{a^3(a^2-1)(a-1)y^2(1+y)}\times$$
$$\begin{vmatrix} 1 & x & 0 & 0 & 0 \\ 1+a^2y, & 1+y, & x & 0 & 1 \\ & (a-1)(a^2-1)y, & 1+ay, & x & a(a^2-1)y \\ & & (1+y)(1+a^3y)a, & 1+a^2y, & a^3(a^2-1)(a-1)y^2 \\ & & & (a^2-1)(a-1)ya^2, & -a^5(a^2-1)(a-1)y^3(1+ay) \end{vmatrix}$$

On this determinant perform the second operation

$$(a-1)\mathrm{col}_5+a^3(a-1)y(1+ay)\mathrm{col}_4-a^2(a-1)y\mathrm{col}_3-a\mathrm{col}_2+\mathrm{col}_1$$

This enables us to remove the factor $(1-ax)$ from the last column and then subtracting the first column from the last we can remove another factor $-(1+ay)$ and get the co-factor in the form

$$\frac{1+a^3y}{y(a-1)}\cdot\begin{vmatrix} 1 & x & \\ 1+a^2y, & 1+y, & 1 \\ & (a-1)(a^2-1)y & a^2(a-1)y \end{vmatrix}$$

On this performing the third operation

$(a^3-1)\text{col}_3-a^2\text{col}_2+\text{col}_1$, we have

$$\frac{1+a^3y}{y(a-1)(a^2-1)}\begin{vmatrix} 1 & x & 1-a^2x \\ 1+a^2y, & 1+y, & 0 \\ & (a-1)(a^2-1)y, & 0 \end{vmatrix}=(1-a^2x)(1+a^2y)\times(1+a^3y)$$

$\therefore$ the continuant of the 6th order$=(1-x)(1-ax)(1-a^2x)(1+y)$
$(1+ay)(1+a^2y)(1+a^3y)$

In the general case if m_k denotes the multiplier of the kth column and l that of the last column, then we have:

In the first operation

$$\begin{cases} m_{2r}=(-1)^r a^{\frac{3r^2-7r+4}{2}}y^{r-1}\{(a^{n-1}-1)(a^{n-2}-1)\dots(a^{n-r+1}-1)\}. \\ \qquad\qquad\{(1+y)(1+ay)\dots(1+a^{r-2}y)\} \\ m_{2r+1}=(-1)^r a^{\frac{3r^2-3r}{2}}y^{r}\{(a^{n-1}-1)(a^{n-2}-1)\dots(a^{n-r}-1)\}. \\ \qquad\qquad\{(1+y)(1+ay)\dots(1+a^{r-1}y)\} \\ l \text{ is governed by these two rules.} \end{cases}$$

In the second operation

$$\begin{cases} m_{2r}=(-1)^r a^{\frac{3r^2-5r+4}{2}}y^{r-1}\{(a^{n-2}-1)(a^{n-3}-1)\dots(a^{n-r}-1)\} \\ \qquad\qquad\{(1+ay)(1+a^2y)\dots(1+a^{r-1}y\} \\ m_{2r+1}=(-1)^r a^{\frac{3r^2-r+2}{2}}y^{r}\{(a^{n-2}-1)(a^{n-3}-1)\dots(a^{n-r-1}-1)\} \\ \qquad\qquad\{(1+ay)(1+a^2y)\dots(1+a^{r-1}y)\} \\ l=a-1. \end{cases}$$

In the third operation

$$\begin{cases} m_{2r}=(-1)^r a^{\frac{3r^2-3r+4}{2}}y^{r-1}\{(a^{n-3}-1)(a^{n-4}-1)\dots(a^{n-r-1}-1)\} \\ \qquad\qquad\{(1+a^2y)(1+a^3y)\dots(1+a^ry)\} \\ m_{2r+1}=(-1)^r a^{\frac{3r^2+r+4}{2}}y^{r}\{(a^{n-3}-1)(a^{n-4}-1)\dots(a^{n-r-2}-1)\} \\ \qquad\qquad\{(1+a^2y)(1+a^3y)\dots(1+a^ry)\} \\ l=a^2-1. \end{cases}$$

In the fourth operation

$$\left\{\begin{aligned} m_{2r} &= (-1)^r a^{\frac{3r^2-r+4}{2}} y^{r-1}\{(a^{n-4}-1)(a^{n-5}-1)\ldots(a^{n-r-2}-1)\} \\ &\qquad \{(1+a^3y)(1+a^4y)\ldots(1+a^{r+1}y)\} \\ m_{2r+1} &= (-1)^r a^{\frac{3r^2+3r+6}{2}} y^r\{(a^{n-4}-1)(a^{n-5}-1)\ldots(a^{n-r-3}-1)\} \\ &\qquad \{(1+a^3y)(1+a^4y)\ldots(1+a^{r+1}y)\} \\ l &= a^3-1. \end{aligned}\right.$$

and so on.

In each of these operations m_1 is always unity. After each operation being performed we shall find a factor of the form $(1-a^r x)$ removable from the last column. Removing this factor and subtracting the first column from the last we shall find another factor of the form $-(1+a^r y)$ removable from the last column. On removing this factor we shall have the co-factor in the form of a determinant on which the next operation is to be performed.

2. $\{x(x+1)(x+2)\ldots(x+r-1)\}$

$$\begin{aligned} &= \{(x-\delta)(x-\delta-1)(x-\delta-2)\ldots(x-\delta-\overline{r-1})\} \\ &+ (\delta+r-1)\,{}^{r}C_1\{(x-\delta)(x-\delta-1)\ldots(x-\delta-\overline{r-2})\} \\ &+ (\delta+r-1)(\delta+r-2)\,{}^{r}C_2\{(x-\delta)(x-\delta-1)\ldots(x-\delta-\overline{r-3})\} \\ &+ \ldots\ldots + \{(\delta+r-1)(\delta+r-2)(\delta+r-3)\ldots(\delta+1)\}\,{}^{r}C_{r-1}(x-\delta) \\ &+ \{(\delta+r-1)(\delta+r-2)\ldots(\delta+1)\delta\} \text{ identically.} \qquad \ldots \quad (1) \end{aligned}$$

Proof. If we substitute any of the values $0, -1, -2, \ldots, -(r-1)$ for x in (1), then by means of difference formulæ we can show that in each case of these substitutions the left-hand-side expression = the right-hand-side expression = 0. Again if $x=\delta$, each of the two expressions is equal to the last term of the right-hand-side expression. Thus for more than r values of x, the equation (1) is satisfied. Hence it is an identity.

Ex. 1. Putting $2x = x+1$ and $2\delta = \delta+1$ in (1) we get as a particular case of the theorem (1), the same identity as given in Art 1 of the first part.

Ex. 2. Putting $2\delta=1$, $r=h$ and $2x=a+2h-1$ in (1) we have

$$\{(a+2h-1)(a+2h+1)(a+2h+3)\ldots\ldots(a+4h-3)\}$$
$$=\{(a+2h-2)(a+2h-4)\ldots(a+2)a\}$$
$$+(2h-1)\overset{h}{C}_1\{(a+2h-2)(a+2h-4)\ldots(a+2)\}$$
$$+(2h-1)(2h-3)\overset{h}{C}_2\{(a+2h-2)(a+2h-4)\cdots(a+4)\}+\ldots\ldots$$
$$+\{(2h-1)(2h-3)\ldots(2h-2k+1)\}\overset{h}{C}_k\{(a+2h-2)(a+2h-4)\ldots(a+2k)\}$$
$$+\ldots\ldots+\{(2h-1)(2h-3)\ldots3\}\overset{h}{C}_{k-1}(a+2h-2)$$
$$+\{(2h-1)(2h-3)\cdots3\cdot1\}\overset{h}{C}_k \qquad \ldots \quad (2)$$

But

$$\{(2h-1)(2h-3)\ldots(2h-2k+1)\}\overset{h}{C}_k$$
$$=\frac{\underline{|h}}{\underline{|h-k}\ \underline{|k}}\{(2h-1)(2h-3)\ldots(2h-2k+1)\}$$
$$=\frac{\{h(h-1)(h-2)\ldots(h-k+1)\}}{\underline{|k}}\{(2h-1)(2h-3)\cdots(2h-2k+1)\}$$
$$=\frac{\{2h(2h-2)(2h-4)\ldots(2h-2k+2)\}}{2\cdot4\cdot6\ldots(2k)}\{(2h-1)(2h-3)\ldots(2h-2k+1)\}$$
$$=\frac{\underline{|2h}}{\underline{|2h-2k}\{2\cdot4\cdot6\ldots(2k)\}}=\{1\cdot3\cdot5\cdot\ldots(2k-1)\}\frac{\underline{|2h}}{\underline{|2h-2k}\ \underline{|2k}}$$
$$=\{1\cdot3\cdot5\cdot\ldots(2k-1)\}\overset{2h}{C}_{2h-2k}$$

Hence from (2) we have

$$\{(a+2h-1)(a+2h+1)\cdots(a+4h-3)\}$$
$$=\{a(a+2)(a+4)\ldots(a+2h-2)\}\overset{2h}{C}_{2h}$$
$$+1\cdot\overset{2h}{C}_{2h-2}\{(a+2)(a+4)\ldots(a+2h-2)\}$$
$$+1\cdot3\overset{2h}{C}_{2h-4}\{(a+4)(a+6)\ldots(a+2h-2)\}+\ldots\ldots$$
$$+\{1\cdot3\cdot5\cdot\ldots(2k-1)\overset{2h}{C}_{2h-2k}\{a+2k)(a+2k+2)\ldots(a+2h-2)\}$$
$$+\ldots\ldots+\{1\cdot3\cdot5\cdot\ldots(2h-3)\}\overset{2h}{C}_2(a+2h-2)+\{1\cdot3\cdot5\cdot\ldots(2h-1)\}\overset{2h}{C}_0$$

which is the same indentity as given in Art. 2 of the first part.

Ex. 3. Similarly by putting $2\delta=3, r=h-1$ and $2x=a+2h-1$ in the theorem (1) we can show that

$$\{(a+2h-1)(a+2h+1)\ldots(a+4h-5)\}=\{a(a+2)\ldots(a+2h-4)\}$$
$$+1\cdot\overset{2h-1}{C}_{2h-3}\{a+2)(a+4)\ldots(a+2h-4)\}$$
$$+1\cdot3\overset{2h-1}{C}_{2h-5}\{(a+4)(a+6)\ldots(a+2h-4)\}$$
$$+1\cdot3\cdot5\overset{2h-1}{C}_{2h-7}\{(a+6)(a+8)\ldots(a+2h-4)\}+\ldots\ldots$$
$$+\{1\cdot3\cdot5\ldots(2h-5)\}\overset{2h-1}{C}_{3}(a+2h-4)+\{1\cdot3\cdot5\ldots(2h-3)\}\overset{2h-1}{C}_{1}$$

This identity has been given in Art. 3 of the first part.

Ex. 4. $$\underline{|n-1}+\underline{|n-2}\overset{n-k}{C}_{2}(a+k-1)+\underline{|n-3}\overset{n-k}{C}_{3}(a+k)(a+k-1)$$
$$+\underline{|n-4}\overset{n-k}{C}_{3}(a+k+1)(a+k)(a+k-1)+\ldots\ldots$$
$$+\underline{|k-1}\{a+n-2)(a+n-3)\ldots(a+k-1)\}\overset{n-k}{C}_{n-k}$$
$$=\underline{|k-1}\{(a+2k-1)(a+2k)(a+2k+1)\ldots(a+n+k-2)\}$$

identically, where n and k are both positive integers and k is less than n.

This identity may be deduced from the theorem (1) by putting $\delta=1-n$, $r=n-k$ and $-x=a+n+k-2$.

Ex. 5. $$\frac{\underline{|n-h-1}}{\underline{|r-1}}\overset{n-r}{C}_{h}\overset{h}{C}_{h}+\frac{\underline{|1}\;\underline{|n-h-2}}{\underline{|r-1}}\overset{n-r}{C}_{h+1}\overset{h+1}{C}_{h}\overset{a+r+h-1}{C}_{1}$$
$$+\frac{\underline{|2}\;\underline{|n-h-3}}{\underline{|r-1}}\overset{n-r}{C}_{h+2}\overset{h+2}{C}_{h}\overset{a+r+h}{C}_{2}+\frac{\underline{|3}\;\underline{|n-h-4}}{\underline{|r-1}}\overset{n-r}{C}_{h+3}\overset{h+3}{C}_{h}\overset{a+r+h+1}{C}_{3}$$
$$+\ldots\ldots+\frac{\underline{|n-r-h-1}\;\underline{|r-2}}{\underline{|r-1}}\overset{n-r}{C}_{n-r-1}\overset{n-r-1}{C}_{h}\overset{a+n-3}{C}_{n-r-h-1}$$
$$+\frac{\underline{|r-1}\;\underline{|n-r-h}}{\underline{|r-1}}\overset{n-r}{C}_{n-r}\overset{n-r}{C}_{h}\overset{a+n-2}{C}_{n-r-h}$$
$$=\overset{n-r}{C}_{h}\{(a+2r+h-1)(a+2r+h)(a+2r+h+1)\ldots(a+n+r-2)\}$$

identically where h is $<n-r$ and $r<n$.

Proof. The left-hand-side expression

$$= {}^{n-r}C_h \left[\frac{|\underline{n-h-1}}{|\underline{r-1}} + \frac{|\underline{1}\ \ |\underline{n-h-2}}{|\underline{r-1}}\ {}^{n-r-h}C_1\ {}^{a+r+h-1}C_1 \right.$$

$$+\frac{|\underline{2}\ \ |\underline{n-h-3}}{|\underline{r-1}}\ {}^{n-r-h}C_2\ {}^{a+r+h}C_2 + \frac{|\underline{3}\ \ |\underline{n-h-4}}{|\underline{r-1}}\ {}^{n-r-h}C_3\ {}^{a+r+h+1}C_3$$

$$+\dots\dots+\frac{|\underline{r-2}\ \ |\underline{n-r-h-1}}{|\underline{r-1}}\ {}^{n-r-h}C_{n-r-h-1}\ {}^{a+n-3}C_{n-1-h-1}$$

$$\left. +\frac{|\underline{n-1}\ \ |\underline{n-r-h}}{|\underline{r-1}}\ {}^{n-r-h}C_{n-1-h}\ {}^{a+n-2}C_{n-r-h} \right]$$

If $k=n-h$, $b=a+h$ and hence $b+k=n+a$, then the left-hand-side expression

$$= {}^{n-r}C_h \left[\frac{|\underline{k-1}}{|\underline{r-1}} + \frac{|\underline{k-2}}{|\underline{r-1}}\ {}^{k-r}C_1\ (b+r-1) + \frac{|\underline{k-3}}{|\underline{r-1}}\ {}^{k-r}C_2\ (b+r)(b+r-1) \right.$$

$$+\frac{|\underline{k-4}}{|\underline{r-1}}\ {}^{k-r}C_3\ (b+r+1)(b+r)(b+r-1)+\dots\dots$$

$$\left. +\frac{|\underline{r-1}}{|\underline{r-1}}\ {}^{k-r}C_{k-r}\{(b+k-2)(b+k-3)\dots(b+r-1)\} \right]$$

$$= {}^{n-r}C_h\ (b+2r-1)(b+2r)(b+2r+1)\dots(b+k+r-2)$$ by example 4.

Hence the left-hand-side expression

$$= {}^{n-r}C_h \{(a+h+2r-1)(a+h+2r)(a+h+2r+1)\dots\dots(a+n+r-2)\}$$

Thus the identity is proved.

3. If $(a, b, x)^r_d$ denote the expression $\{a(a+b)(a+2b)$

$$\ldots(a+\overline{r-1}b)\}-{}^rC_1\{(a-d)(a+b)(a+2b)\ldots(a+\overline{r-1}b)\}x$$

$$+{}^rC_2\{(a-d)(a+b-d)(a+2b)(a+3b)\ldots(a+\overline{r-1}b)\}x^2$$

$$-{}^rC_3\{(a-d)(a+b-d)(a+2b-d)(a+3b)\ldots(a+\overline{r-1}b)\}x^3+\ldots$$

$$+(-1)^r\,{}^rC_r\{(a-d)(a+b-d)\ldots(a+\overline{r-1}b-d)\}x^r.$$

then

$$(a, b, x)^r_b-x(a+b, b, x)^r_b=\{a(a+b)(a+2b)\ldots(a+\overline{r-1}b)\}.$$

$$\times(1-x)^{r+1} \text{ identically} \qquad \ldots \quad (3)$$

For the coefficient of x^h in $(a, b, x)^r_b$ is $(-1)^h\,{}^rC_h\{(a-b)a(a+b)$

$$(a+2b)\ldots(a+\overline{h-2}b)(a+hb)(a+\overline{h+1}b)\ldots(a+\overline{r-1}b)\}$$ and the

coefficient of x^h in $-(a+b, b, x)^r_b x$ is $(-1)^h\,{}^rC_{h-1}\{a(a+b)(a+2b)$

$$\ldots(a+\overline{h-2}b)(a+hb)(a+\overline{h+1}b)\ldots(a+rb)\}$$

∴ the coefficient of x^h in the left-hand-side expression of (3) is

$$(-1)^h\{a(a+b)(a+2b)\ldots(a+\overline{h-2}b)(a+hb)(a+\overline{h+1}b)\ldots(a+\overline{r-1}b)\}.$$

$$\times[{}^rC_h(a-b)+{}^rC_{h-1}(a+rb)]=(-1)^h\{a(a+b)(a+2b)$$

$$\ldots(a+\overline{h-2}b)(a+hb)(a+\overline{h+1}b)\ldots(a+\overline{r-1}b)\}\,{}^{r+1}C_h(a+\overline{h-1}b)$$

$$=(-1)^h\{a(a+b)(a+2b)\ldots(a+\overline{r-1}b)\}\,{}^{r+1}C_h.$$

$$\therefore\ (a, b, x)^r_b-(a+b, b, x)^r_b x=\{a(a+b)(a+2b)$$

$$\ldots(a+\overline{r-1}b)\}\{1-{}^{r+1}C_1x+{}^{r+1}C_2x^2-\ldots+(-1)^{r+1}\,{}^{r+1}C_{r+1}x^{r+1}\}$$

$$=\{a(a+b)(a+2b)\ldots(a+\overline{r-1}b)\}(1-x)^{r+1}.$$

4. $(a+rb)\overset{r}{(a,\ b,\ x)}_k - a\overset{r}{(a+b,\ b,\ x)}_k - rb^2x\overset{r-1}{(a+2b,\ b,\ x)}_{k-1}=0$

identically... (4)

For, in the left-hand-side expression of (4), the coefficient of x^k is

$$(-1)^h\ {}^rC_k\{(a-b)(a+b)\dots(a+\overline{h-2}b)(a+hb)\dots(a+rb)\}$$

$$-(-1)^h\ {}^rC_k\{a^2(a+b)(a+2b)\cdots(a+\overline{h-1}b)(a+\overline{h+1}b)$$

$$\dots(a+rb)\}+(-1)^k\ {}^{r-1}C_{k-1}rb^2\{a(a+b)\dots(a+\overline{h-2}b)(a+\overline{h+1}b)\dots(a+rb)\}$$

$$=(-1)^h\ \{a(a+b)(a+2b)\dots(a+\overline{h-2}b)\,(a+\overline{h+1}b)\dots(a+rb)\}$$

$$\times[{}^rC_k(a-b)(a+hb)-{}^rC_ka(a+\overline{h-1}b)+b^2r\ {}^{r-1}C_{k-1}]$$

$$=(-1)^h\ \{a(a+b)(a+2b)\dots(a+\overline{h-2}b)(a+\overline{h+1}b)\dots(a+rb)\}$$

${}^rC_k[(a-b)(a+hb)-a(a+\overline{h-1}b)+hb^2]=0$ for the expression within the brackets [ ] is zero. Hence the theorem is proved.

5. If ${}^nS_r$ denote the sum of the products of the n factors, $1,\ a,\ a^2,\cdots a^{n-1}$ taken r of them at a time and if $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the product $(a^n-1)(a^{n-1}-1)\cdots(a^r-1)$ then

$$\begin{bmatrix} h-1 \\ r \end{bmatrix}-\frac{1}{a^k}(a^{k-r}-1)\begin{bmatrix} h-2 \\ r \end{bmatrix}{}^{r+k}S_1 + \frac{1}{a^{2k}}\begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-3 \\ r \end{bmatrix}{}^{r+k}S_2 - \frac{1}{a^{3k}}\begin{bmatrix} h-r \\ h-r-2 \end{bmatrix}\begin{bmatrix} h-4 \\ r \end{bmatrix}{}^{r+k}S_3+\dots$$

$$+(-1)^{k-r}\ \frac{1}{a^{k(h-r)}}\begin{bmatrix} h-r \\ 1 \end{bmatrix}{}^{r+k}S_{k-r}=0$$

where h, r and k are all positive integers k varying from 0 to $h-r-1$.[1]

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[1] *Note:—If n is less than r, ${}^nS_r$ is to be taken as zero but $\begin{bmatrix} n \\ r \end{bmatrix}$ as unity. If $n=r$ then $\begin{bmatrix} n \\ r \end{bmatrix}$ denotes a single factor viz., a^n-1.*

(*i*) Let us take the series u_1, u_2, u_3, u_4,...and obtain from it another series by subtracting each term from the term which immediately precedes it. The series $u_1-\overset{1}{S}_1\,u_2, u_2-\overset{1}{S}_1\,u_3, u_3-\overset{1}{S}_1\,u_4,\cdots$, thus found, may be called the series of the first order of differences and let this series be denoted by Δ_1. Multiply each term, of Δ_1, by a and subtract the product from the term which immediately precedes it, then we get the series of the second order of differences *viz.*,

$$u_1-\overset{2}{S}_1u_2+\overset{2}{S}_2u_3,\ u_2-\overset{2}{S}_1u_3+\overset{2}{S}_2u_4,\ u_3-\overset{2}{S}_1u_4+\overset{2}{S}_2u_5\ldots$$ which may be denoted by Δ_2.

Similarly we are to get Δ_3 from Δ_2 by using a^2 as a multiplier
$\cdots\Delta_4\cdots$ $\Delta_3\cdots$ $a^3\cdots$.

and so on.

Here we observe that some formula holds in the case of each term of any of the series Δ_1 and Δ_2. Let us assume that this formula holds in the case of Δ_{r-1} *i.e.*, suppose Δ_{r-1} is

$$u_1-\overset{r-1}{S}_1\,u_2+\overset{r-1}{S}_2\,u_3-\ldots+(-1)^{r-1}\overset{r-1}{S}_{r-1}u_r,\ u_2-\overset{r-1}{S}_1\,u_3$$
$$+\ldots(-1)^{r-1}\overset{r-1}{S}_{r-1}u_{r+1},\ldots$$

Then by multiplying each term, by a^{r-1} and subtracting the product from the term that immediately precedes it, we get the r^{th} order of differences *viz.*,

$$u_1-\overset{r}{S}_1u_2+\overset{r}{S}_2u_3-\ldots(-1)^r\overset{r}{S}_ru_{r+1},u_2-\overset{r}{S}_1u_3+\overset{r}{S}_2u_4$$
$$-\ldots(-1)^r\overset{r}{S}_ru_{r+2},\ldots\left[\text{for } \overset{r-1}{S}_n+\ a^{r-1}\ {}^{r-1}S_{n-1}=\overset{r}{S}_n\ \right].$$

Thus if the formula holds in the case of Δ_{r-1}, it also holds in the case of Δ_r. But it holds in the case of Δ_1, and Δ_2 and hence it holds universally,

(*ii*). If $\overset{n}{A}_r$ donote the sum of the products of n factors $a, a,^2, a^3\ldots a^n$ taken r of them at a time then it is evident that ${}^rA_k+{}^rA_{k-1}={}^rA_k+1\cdot{}^rA_{k-1}={}^{r+1}S_k$ and since every term of $\overset{h}{S}_r$ is the product of r factors and each of these factors, multiplied by a, gives the corresponding factor in the corresponding term of $\overset{h}{A}_r$, we have $a^r\overset{h}{S}_r=\overset{h}{A}_r$.

(*iii*). Let us take the series Δ_r *viz*

$$u_1 - {}'S_1u_2 + \overset{r}{S}_2u_3 - \ldots + (-1)^r\overset{r}{S}_ru_{r+1}, u_2 - \overset{r}{S}_1u_3 + \overset{r}{S}_2u_4$$
$$- \ldots + (-1)^r S_r u_{r+2}, \ldots$$

Multiply each term, of Δ_r by $\frac{1}{a}$ and subtract the product from the term which immediately precedes it Let the series, thus found, be denoted by D_1, : Then the first term of D_1 is

$$u_1 - \frac{1}{a}(a\overset{r}{S}_1+1)u_2 + \frac{1}{a^2}(a^2\overset{r}{S}_2 + a\overset{r}{S}_1)u_3 - \ldots$$

$$+(-1)^r\frac{1}{a^r}(a^r\overset{r}{S}_r + a^{r-1}\overset{r}{S}_{r-1})u_{r+1} + (-1)^{r+1}\frac{1}{a^{r+1}}\,a^r\overset{r}{S}_ru_{r+2}$$

$$=u_1 - \frac{1}{a}\,(\overset{r}{A}_1+1)u_2 + \frac{1}{a^2}(\overset{r}{A}_2+\overset{r}{A}_1)u_3\ldots$$

$$+(-1)^r\frac{1}{a^r}(\overset{r}{A}_r+\overset{r}{A}_{r-1})u_{r+1} + (-1)^{r+1}\frac{1}{a^{r+1}}\overset{r+1}{S}_{r+1}u_{r+2} \text{ by (ii)}$$

$$=u_1 - \frac{1}{a}\overset{r+1}{S}_1\,u_2 + \frac{1}{a^2}\overset{r+1}{S}_2\,u_3\ldots$$

$$+(-1)^r\frac{1}{a^r}\overset{r+1}{S}_r\,u_{r+1} + (-1)^{r+1}\frac{1}{a^{r+1}}\overset{r+1}{S}_{r+1}u_{r+2} \text{ by (ii)}$$

Similarly the other terms of D_1 may be obtained.

We can similorly show that if the series obtained from D_1, by using $\frac{1}{a^2}$ as a multiplier, be denoted by D_2, then D_2 is

$$u_1 - \frac{1}{a^2}\,\overset{r+2}{S}_1u_2 + \frac{1}{a^4}\,\overset{r+2}{S}_2u_3 - \ldots + (-1)^{r+2}\frac{1}{a^{2(r+2)}}\,\overset{r+2}{S}_{r+2}u_{r+3}, u_2 - \frac{1}{a^2}\overset{r+2}{S}_1\,u_3$$

$$+\ldots+(-1)^{r+2}\frac{1}{a^{2(r+2)}}\overset{r+2}{S}_{r+2}\,u_{r+4},\ldots$$

Thus D_3, $D_4\ldots$ may be obtained by using $\frac{1}{a^3}$, $\frac{1}{a^4}\ldots$ as multipliers.

Then by the method of induction we can show that D_k is the series

$$u_1 - \frac{1}{a^k}\,{}^{r+k}S_1 u_2 + \frac{1}{a^{2k}}\,{}^{r+k}S_2 u_3 - \dots + (-1)^{r+k}\frac{1}{a^{k(r+k)}}\,{}^{r+k}S_{r+k}\,u_{r+k+1},$$

$$u_2 - \frac{1}{a^k}\,{}^{r+k}S_1\,u_3 + \dots + (-1)^{r+k}\frac{1}{a^{k(r+k)}}\,{}^{r+k}S_{r+k}u_{r+k+2},\dots$$

(iv). Let us now take the seaies

$$\begin{bmatrix} h-1 \\ r \end{bmatrix}, \left(a^{h-r}-1\right)\begin{bmatrix} h-2 \\ r \end{bmatrix}, \begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-3 \\ r \end{bmatrix},$$

$$\begin{bmatrix} h-r \\ h-r-2 \end{bmatrix}\begin{bmatrix} h-4 \\ r \end{bmatrix}, \dots\dots \begin{bmatrix} h-r \\ 2 \end{bmatrix}(a^r-1)$$

$$\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \quad 0, \quad 0, \quad 0, \quad 0\dots$$

Then

$$\Delta_1 \text{ is } a^{h-r}\begin{bmatrix} h-2 \\ r-2 \end{bmatrix}, \; a^{h-r-1}\left(a^{h-r}-1\right)\begin{bmatrix} h-3 \\ r-1 \end{bmatrix},$$

$$a^{h-r-2}\begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-4 \\ r-1 \end{bmatrix}\dots a\begin{bmatrix} h-r \\ 2 \end{bmatrix}(a^{r-1}-1),$$

$$\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \quad 0, \quad 0, \quad 0\dots$$

$$\Delta_2 \text{ is } a^{2(h-r)}\begin{bmatrix} h-3 \\ r-2 \end{bmatrix}, \; a^{2(h-r-1)}\left(a^{h-r}-1\right)\begin{bmatrix} h-4 \\ r-2 \end{bmatrix},$$

$$a^{2(h-r-2)}\begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-5 \\ r-2 \end{bmatrix}\dots a^2\begin{bmatrix} h-r \\ 2 \end{bmatrix}(a^{r-2}-1),$$

$$\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \quad 0. \quad 0, \quad 0\dots$$

$$\dots \qquad \dots \qquad \dots$$

$$\Delta_{r-1} \text{ is } a^{(r-1)(h-r)}\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \; a^{(r-1)(h-r-1)}\begin{bmatrix} h-r \\ 1 \end{bmatrix},$$

$$a^{(r-1)(h-r-2)}\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \dots a^{r-1}\begin{bmatrix} h-r \\ 1 \end{bmatrix}, \begin{bmatrix} h-r \\ 1 \end{bmatrix}, \quad 0, \quad 0, \quad 0\dots$$

$$\Delta_r \text{ is } \quad 0, \quad 0; \quad 0\dots\dots \quad 0, \quad \begin{bmatrix} h-r \\ 1 \end{bmatrix}, \quad 0, \quad 0, \quad 0\dots$$

Hence all the terms except $h-r+1^{th}$ of Δ_r, except $h-r+1^{th}$ and $h-r^{th}$ of D_1, except $h-r+1^{th}$, $h-r^{th}$ and $h-r-1^{th}$ of D_2 etc are zero. In the case of D_{h-r-1}, the first term is zero but all the terms from the 2^{nd} to the $h-r+1^{th}$ do not vanish. Thus the first term of each of $\Delta_r, D_1, D_2, \ldots D_{h-r-1}$ is zero. Hence by (*i*) and (*iii*) we have

$$\begin{bmatrix} h-1 \\ r \end{bmatrix} - \frac{1}{a^k}\left(a^{h-r}-1\right)\begin{bmatrix} h-2 \\ r \end{bmatrix}\overset{r+k}{S}_1 + \frac{1}{a^{2k}}\begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-3 \\ r \end{bmatrix}\overset{r+k}{S}_2$$

$$-\ldots+(-1)^{h-r}\frac{1}{a^{k(h-r)}}\begin{bmatrix} h-r \\ 1 \end{bmatrix}\overset{r+k}{S}_{h-r}=0$$

where k varies from 0 to $h-r-1$.

It is interesting to note in this connection that if $\overset{n}{S}_r$ denotes the sum of the products of n factors $a_1, a_2, a_3 \ldots a_n$ taken r of them at a time where a's are all arbitrary, then using a_1, a_2, a_3 etc. as successive multipliers it can eassily be shown by induction that the r^{th} order of differences is the series

$$u_1-\overset{r}{S}_1 u_2+\overset{r}{S}_2 u_3-\ldots(-1)^{r}\overset{r}{S}_r u_{r+1},\ u_2-\overset{r}{S}_1 u_3+\overset{r}{S}_2 u_4$$

$$-\ldots,(-1)^{r}\overset{r}{S}_r u_{r+2},\ldots\left[\text{for } \overset{r}{S}_k+a_{r+1}\overset{r}{S}_{k-1}=\overset{r+1}{S}_k\right]$$

6.
$$\overset{k+r}{S}_k=\frac{\begin{bmatrix} k+r \\ 1 \end{bmatrix}}{\begin{bmatrix} r \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}}\,{}^{k}S_k \text{ identically.}$$

Let T_k denote the series $\overset{h}{S}_k,\ \overset{h+1}{S}_k,\ \overset{h+2}{S}_k \ldots$, then we find by trial that the theorem holds good in the case of each term of the series T_1. Let us assume that it holds in the case of the series T_{k-1}.

The first term of T_k is $\overset{k}{S}_k$.

The second term of T_k is $\overset{k+1}{S}_k=\overset{k}{S}_k+a^k\overset{k}{S}_{k-1}$

$$=\overset{k}{S}_k+a^k\,\frac{a^k-1}{a-1}\,{}^{k-1}S_{k-1}=\overset{k}{S}_k\left(1+a\,\frac{a^k-1}{a-1}\right)=\frac{a^{k+1}-1}{a-1}\overset{k}{S}_k$$

$$=\frac{\begin{bmatrix} k+1 \\ 1 \end{bmatrix}}{\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\overset{k}{S}_k.$$

Thus we see that the theorem holds in the case of the first two terms of T_k. Suppose it holds good in the case of ${}^{k+r-1}S_k$

$$\text{But}\qquad \overset{k+r}{S_k} = {}^{k+r-1}S_k + a^{k+r-1}\,{}^{k+r-1}S_{k-1} = \frac{\begin{bmatrix} k+r-1 \\ k+1 \end{bmatrix}}{\begin{bmatrix} r-1 \\ 1 \end{bmatrix}} \overset{k}{S_k}$$

$$+a^{k+r-1}\frac{\begin{bmatrix} k+r-1 \\ k \end{bmatrix}}{\begin{bmatrix} r \\ 1 \end{bmatrix}}\,{}^{k-1}S_{k-1} = \frac{\begin{bmatrix} k+r-1 \\ k+1 \end{bmatrix}}{\begin{bmatrix} r-1 \\ 1 \end{bmatrix}}$$

$$\times\left\{\overset{k}{S_k} + a^{k+r-1}\frac{a^k-1}{a^r-1}\,{}^{k-1}S_{k-1}\right\} = \frac{\begin{bmatrix} k+r-1 \\ k+1 \end{bmatrix}}{\begin{bmatrix} r-1 \\ 1 \end{bmatrix}} \overset{k}{S_k}\left\{1+a^r\frac{a^k-1}{a^r-1}\right\}$$

$$= \frac{\begin{bmatrix} k+r \\ k+1 \end{bmatrix}}{\begin{bmatrix} r \\ 1 \end{bmatrix}} \overset{k}{S_k} = \frac{\begin{bmatrix} k+r \\ 1 \end{bmatrix}}{\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} r \\ 1 \end{bmatrix}}.$$

Hence if the theorem is true for $r-1^{th}$ term of the series T_k, it is also true for the r^{th} term of the same series. But it is true for the first two terms of the series T_k. So it holds in the case of each term of the series T_k. Thus we see that if it holds in the case of the series T_{k-1}, it also holds in the case of the series T_k. But it holds in the case of T_1. Hence it holds universally.

7.
$$1-\frac{a^r\left(a^{n-k-r}-1\right)\left(1+a^{r+k-1}y\right)}{(a-1)\left(a^{n-k-1}-1\right)}$$

$$+\frac{a^{2r}\begin{bmatrix} n-k-r \\ n-k-r-1 \end{bmatrix}\left(1+a^{r+k-1}y\right)\left(1+a^{r+k}y\right)}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ n-k-2 \end{bmatrix}}$$

$$-\frac{a^{3r}\begin{bmatrix} n-k-r \\ n-k-r-2 \end{bmatrix}\left\{\left(1+a^{r+k-1}y\right)\left(1+a^{r+k}y\right)\left(1+a^{r+k+1}y\right)\right\}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ n-k-3 \end{bmatrix}}$$

$$+\dots+(-1)^{n-k-r}a^{(n-k-r)r}$$

$$\times\frac{\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}\left\{\left(1+a^{r+k-1}y\right)\left(1+a^{r+k}y\right)\cdots\left(1+a^{n-2}y\right)\right\}}{\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ r \end{bmatrix}}$$

$$=(-1)^{n-k-r}\frac{\left\{\left(1+a^{2r+k-1}y\right)\left(1+a^{2r+k}y\right)\cdots\left(1+a^{n+r-2}y\right)\right\}}{\begin{bmatrix} n-k-1 \\ r \end{bmatrix}}$$

identically.

Proof. If $y=-\frac{1}{a^{r+k-1}}$, each of the two expressions of the above is equal to unity.

If $y=-\frac{1}{a^{2r+k-1}}$, the right-hand-side expression$=0$ and the left-hand-side expression

$$=1-\frac{\left(a^{n-k-r}-1\right)\left(a^{r}-1\right)}{(a-1)\left(a^{n-k-1}-1\right)}+\frac{a\begin{bmatrix} n-k-r \\ n-k-r-1 \end{bmatrix}\begin{bmatrix} r \\ r-1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ n-k-2 \end{bmatrix}}$$

$$\frac{a^{3}\begin{bmatrix} n-k-r \\ n-k-r-2 \end{bmatrix}\begin{bmatrix} r \\ r-2 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ n-k-3 \end{bmatrix}}+\cdots$$

$$+(-1)^{n-k-r}\frac{a^{\frac{1}{2}(n-k-r)(n-k-r-1)}\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}\begin{bmatrix} r \\ 2r+k-n+1 \end{bmatrix}}{\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}\begin{bmatrix} n-k-1 \\ r \end{bmatrix}}$$

$$=1-\frac{a^{n-k-r}-1}{a^{n-k-1}-1}\,{}^{r}S_{1}+\frac{\begin{bmatrix} n-k-r \\ n-k-r-1 \end{bmatrix}}{\begin{bmatrix} n-k-1 \\ n-k-2 \end{bmatrix}}\,{}^{r}S_{2}-\cdots$$

$$+(-1)^{n-k-r}\frac{\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}}{\begin{bmatrix} n-k-1 \\ r \end{bmatrix}}\,{}^{r}S_{n-k-r}.$$

by Art 6

$$=\frac{1}{\begin{bmatrix} n-k-1 \\ r \end{bmatrix}}\left\{\begin{bmatrix} n-k-1 \\ r \end{bmatrix}^r S_{n-k-r}\right.$$

$$-\begin{bmatrix} n-k-2 \\ r \end{bmatrix}\left(a^{n-k-r}-1\right)^r S_1+\begin{bmatrix} n-k-3 \\ r \end{bmatrix}\begin{bmatrix} n-k-r \\ n-k-r-1 \end{bmatrix}^r S_2$$

$$\left.-\ldots+(-1)^{n-k-r}\begin{bmatrix} n-k-r \\ 1 \end{bmatrix}^r S_{n-k-r}\right\}=0$$

for, if we put $n-k=h$, the expression within the brackets becomes

$$\begin{bmatrix} h-1 \\ r \end{bmatrix}-\left(a^{h-r}-1\right)\begin{bmatrix} h-2 \\ r \end{bmatrix}^r S_1+\begin{bmatrix} h-r \\ h-r-1 \end{bmatrix}\begin{bmatrix} h-3 \\ r \end{bmatrix}^r S_2-\ldots$$

$$+(-1)^{h-r}\begin{bmatrix} h-r \\ 1 \end{bmatrix}^r S_{h-r}$$

which is zero by Art 5.[1]

Similarly by means of the theorem given in Art 5, we can show that both the expressions of the equation are zero, when we substitute any of the values $-\frac{1}{a^{2r+k}}, -\frac{1}{a^{2r+k+1}}, \ldots -\frac{1}{a^{r-2}}$ for y in the equation Thus for $n-k-r+1$ values of y the equation is satisfied. Hence it is an identity.

As source of references is too difficult to be available here, so if any of the above results have been discovered by other mathematicians, we shall be very glad to mention their names in proper places.

[1] Bull. Cal. Math. Soc., Vol. XIV, No. 2.

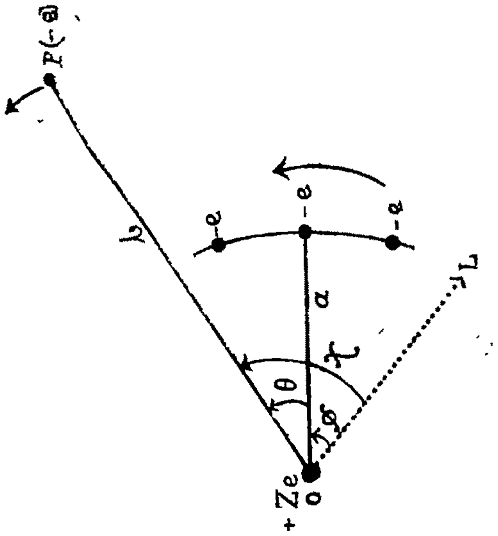

Fig.

[Illustrating inter-atomic motions in K. Basu's Paper, pp. 116-17]

3

ON THE PERTURBATIONS OF THE ORBIT OF THE VALENCY-ELECTRON IN THE GENERALIZED HYDROGEN-UNLIKE ATOM (A)

BY

K. BASU.

I

INTRODUCTION.

According to the modern theories of atomic structure, the atom consists of a nuclear positive charge N*e*, with N electrons rotating round it in different successive shells (N=atomic number). Of late years, attempt has been made to explain the spectral lines as well as the chemical properties of the atom on a dynamical quantum theory of the orbital motion of the electrons. The first attempt in this direction was made by Bohr[1]; by combining the quantum theory of energy exchanges with the nuclear theory of the atom, Bohr was able to explain very successfully the spectral series of hydrogen and ionized helium (He^+).

Bohr's method was generalized by Sommerfeld[2] in a remarkable series of papers. With the aid of the generalized theory of quantum vibration, Sommerfeld succeeded in explaining in a qualitative manner the spectral series of alkalies and ionized alkaline earths, and in laying down certain general rules for the elucidation of the spectra of elements. Further progress in this direction is hampered by our inability to cope with the time-honoured problem of three bodies.

The problem is to find out the motion of any one of the electrons in the combined field of the nucleus and the other electrons according to quantum-mechanics. When the electron happens to be the outermost valency electron, the solution of its motion would provide us with the key to the explanation of its visible spectra. If it happens to be any one of the inner electrons the solution would enable us to explain the K—, L—, M— radiations in the X-ray region.[1]

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[1] Phil. Mag. July, 1913 *et. seqq.*

[2] Sommerfeld "*Atombau und Spektrallinien.*" Chap, 4.

Since an exact solution is not yet in sight, attempts have been made to obtain approximate solutions. Thus Lande[2] and Bohr[3] have tried to tackle helium ($N=2$), Sommerfeld[4] tried to tackle the general case of motion of the outer electron, assuming the total charge of the electrons to be equally distributed in a ring of radius a. But as we know from other sources of evidence, this is far removed from the actual state of affairs. The electrons are arranged in different shells, containing 2, 8, 8, 18, 18, 32, ... electrons which move according to definite quantum-conditions.[5] The problem is therefore to find out (I) The electrical field due to electrons moving in definite shells about the nucleus, (2) to investigate the motion of the outer electron in the combined field.

In the following I have assumed that n electrons, situated at the corners of a regular polygen of n-sides are rotating with angular velocity ω about the nucleus. The general field due to such a ring being found, we can obtain the total field by simple addition. The range of validity of Sommerfeld's assumption has also been investigated. And as a matter of fact, it is very probable that the outer electron cannot describe the same circular orbit permanently under the action of n rapidly moving electrons, on the contrary, it may suffer periodical perturbation, the present attempt aims at determining the perturbed orbit of such an atom conventionally known as hydrogen-unlike (*Wasserstoffunähnlich*).

The ring configuration having $Z-k$ electrons (Z=atomic number) was tackled by Sommerfeld[6] on an assumption of sufficient quickness (*hinreichend rasch*) of revolution of the electrons and he calculated the energy function by a method of approximations, in terms of quantum numbers, which can be utilized to frame the *Haupt*, *Neben* and Bergmann series formulae. The type of such formulae is quite different from that of the hydrogen atom, so the name "hydrogen-unlike atoms" is prescribed to signify another type of series formulae. Such ring configurations have been applied provisionally with apparent success to a wide range of phenomena, notably, the theoretical derivation of Ritz formula and *a fortiori* of the Balmer and Rydberg ones, and the computation of X-ray frequencies.

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[1] Kossel, "*Zs. f. Physik.*," Vol. 2, p. 470; Wentzel. "*Zs. f. Physik.*" Vol. 8. p. 85; Coster. "Phil. Mag.," 1922.

[2] Lande, "*Phys. Zs.*." 1921, p. 114.

[3] Bohr, "*Zs. f. Physik.*," Vol. 9. pp. 1-67.

[4] Sommerfeld. "*Atombau*. third ed.," Anhang, p. 721.

[5] Loring, "Atomic Theories."

[6] '*Atombau und Spektrallinien*' *Zweite Auflage*, Braunschweig, 1921, *Zusätze und Ergänzungen*, § 10 pp. 506-14.

The additional field (*Zusatzfeld*) of Sommerfeld is modified in this paper to fit in with the assumption that the angular velocity of the ring electrons will not be indefinitely large in comparison with that of the valency electron, provided that under certain legitimate limitations the ring configuration as postulated is a stable one[1], and the ionization potentials found out theoretically on such a basis is in agreement with experimental facts.

In fact the term *generalized* is appended to signify an electrically neutral atom, although the present problem is applicable quite well to ionized atoms, and without loss of generality to all heavy atoms ionized to have a single valency electron, since the ring next to the latter has the most important bearing on its motion than other interior rings. Although, as a matter of fact, the theory of ring configuration of atomic systems is losing much of its interests and Bohr (*loc-cit*) conceives of separate orbital configurations for each electron still it is quite apropos of the time which has hardly ever any permanency of theoretical grounds.

II

DETERMINATION OF POTENTIAL FUNCTION.

(*a*) *By the Method of Zonal Harmonics.*

Suppose generally there are n electrons situated at the corners of a regular n-gon, which is rotating with the velocity ω. We wish to find out the potential at an external point (r,θ) having a charge $-e$, the initial line passing through the centre and a particular electron on the ring.

We have

$$V = -\frac{Ze^2}{r} + e^2\left(\frac{1}{r_1} + \frac{1}{r_1} + \dots + \frac{1}{r_n}\right)$$

where $r_1, r_2, \dots r_n$ are the distances of the electrons from (r,θ), the *Aufpunkt*, and $+Ze$ the central charge.

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[1] The stability will be discussed in paper B in the next issue of this *Bulletin*.

Now

$$r_1^2=r^2+a^2-2ra\cos\theta$$

$$r_2^2=r^2+a^2-2ra\cos(\theta+\alpha),\ \text{etc.},$$

$$r_s^2=r^2+a^2-2ra\cos(\theta+\overline{s-1}\alpha),\ \text{where } \alpha=\frac{2\pi}{n}.$$

We have

$$\frac{1}{r_s}=\frac{1}{r}\left(1+\frac{a}{r}P_1(\theta+\overline{s-1}\alpha)+\ldots+\left(\frac{a}{r}\right)^m P_m(\theta+\overline{s-1}\alpha)+\ldots\right).$$

For brevity, let us denote

$$P_m(\theta+\overline{s-1}\alpha)\ \text{by}\ P_{m,s-1}.$$

Then

$$V=-\frac{Ze^2}{r}+\frac{e^2}{r}\left[1+\left(\frac{a}{r}\right)P_{1,0}+\left(\frac{a}{r}\right)^2 P_{2,0}+\ldots\right]$$

$$+\frac{e^2}{r}\left[1+\left(\frac{a}{r}\right)P_{1,1}+\left(\frac{a}{r}\right)^2 P_{2,1}+\ldots\right]+\text{etc.},$$

$$+\frac{e^2}{r}\left[1+\left(\frac{a}{r}\right)P_{1,n-1}+\left(\frac{a}{r}\right)^2 P_{2,n-1}+\ldots\right].$$

That is

$$V=-\frac{Ze^2}{r}+\frac{e^2}{r}\left[n+\left(\frac{a}{r}\right)S_1^n+\left(\frac{a}{r}\right)^2 S_2^n+\ldots\right],\ \text{if}$$

$$S_m^n=P_{m,0}+P_{m,1}+P_{m,2}+\ldots+P_{m,n-1}.$$

We know

$$P_{m,0}=\frac{1\cdot3\cdot5\ldots2m-1}{2\cdot4\cdot6\ldots2m}\left[2\cos m\theta+2\frac{1\cdot m}{1\cdot 2m-1}\cos(m-2)\theta\right.$$

$$\left.+2\cdot\frac{1\cdot3\cdot m(m-1)}{1\cdot2\cdot(2m-1)(2m-3)}\cos(m-4)\theta+\ldots\right],$$

similar values for $P_{m,1}$, $P_{m,2}$, etc. [See Byerly's Spherical Harmonics p. 159.]

Whence

$$S^n_m = \frac{1\cdot3\cdot5\ldots2m-1}{2\cdot4\cdot6\ldots2m}\,[2\{\cos m\theta+\cos m(\theta+\alpha)+\cos m(\theta+2\alpha)$$

$$+\ldots+\cos m(\theta+\overline{n-1}\alpha)\}+2\cdot\frac{1\cdot m}{1\cdot2m-1}\{\cos(m-2)\theta$$

$$+\cos(m-2)(\theta+\alpha)+\ldots+\cos(m-2)(\theta+\overline{n-1}\alpha)\}$$

$$+2\cdot\frac{1\cdot3\cdot m(m-1)}{1\cdot2(2m-1)(2m-3)}\{\cos(m-4)\theta+\cos(m-4)(\theta+\alpha)$$

$$+\ldots+\cos(m-4)(\theta+\overline{n-1}\alpha)\}+\cdots].$$

Put

$$C^n_s(\theta)=\cos s\theta+\cos s(\theta+\alpha)+\ldots+\cos s(\theta+\overline{n-1}\alpha)\cdot$$

$$(s=m,\ m-2,\ m-4,\ \text{etc.})$$

$$\therefore C^n_s(\theta)=\cos\left\{s\theta+(n-1)\frac{s\alpha}{2}\right\}\sin\frac{n}{2}s\alpha/\sin\frac{s\alpha}{2}.$$

$$=\cos\{s\theta+\frac{n-1}{n}s\pi\}\sin s\pi/\sin\frac{s\pi}{n}.$$

$$=0,\ (s\neq n).$$

$$C^n_n(\theta)=\cos\{n\theta+(n-1)\pi\}\sin n\pi/\sin\pi.$$

Now, if n odd, $\sin n\pi/\sin\pi=n$; if n even, $\sin n\pi/\sin\pi=-n$; therefore $C^n_n(\theta)=\pm n\cos\{n\theta+(n-1)\pi\}$, according as n is odd or even $=n\cos n\theta$, whether n be even or odd; and $C^n_{2n}(\theta)=\cos\{2n\theta+(n-1)\pi\}\sin 2n\pi/\sin 2\pi=n\cos 2n\theta$, whether n be even or odd; and so on for $C^n_{rn}(\theta)$, $(r=1,2,\ldots ad inf)$.

When m even,

$$S^n_m=n\left\{\frac{1\cdot3\cdot5\ldots m-1}{2\cdot4\cdot6\ldots m}\right\}^2;\ \text{(Byerly, *loc. cit.*)}$$

when m odd,

$$S^n_m=0,\ (m\neq n)$$

And $$S^n_n = \frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}\cdot 2C^n_n(\theta)$$

$$=\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}\cdot 2n\cos n\theta\,;$$

$$S^n_{n+2}=\frac{1\cdot3\cdot5\ldots2n+3}{2\cdot4\cdot6\ldots2n+4}\left[2C^n_{n+2}(\theta)+2\cdot\frac{1\cdot n+2}{1\cdot2n+3}C^n_n(\theta)\right]$$

$$=\frac{1\cdot3\cdot5\ldots2n+3}{2\cdot4\cdot6\ldots2n+4}\cdot2\cdot\frac{1\cdot n+2}{1\cdot2n+3}n\cos n\theta,$$

all others except $C^n_n(\theta)$ vanish

($\because C^m_n(\theta)=0$, $m\neq n$ as shown *ante*).

$$S^n_{n+1}=\frac{1\cdot3\cdot5\ldots2n+7}{2\cdot4\cdot6\ldots2n+8}[2C^n_{n+4}(\theta)+2\cdot\frac{1\cdot n+4}{1\cdot2n+7}C^n_{n+2}(\theta)$$

$$+2\cdot\frac{1\cdot3}{1\cdot2}\cdot\frac{(n+4)(n+3)}{(2n+7)(2n+5)}\cdot C^n_n(\theta)+\ldots]$$

$$=\frac{1\cdot3\cdot5\ldots2n+7}{2\cdot4\cdot6\ldots2n+8}\cdot2\cdot\frac{1\cdot3}{1\cdot2}\cdot\frac{(n+4)(n+3)}{(2n+7)(2n+5)}n\cos n\theta\,;$$

all other terms contributing nothing, excepting $C^n_n(\theta)$; and so on.

Hence

$$V=-\frac{Ze^2}{r}+\frac{ne^2}{r}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{a}{r}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{a}{r}\right)^4\right.$$

$$\left.+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\left(\frac{a}{r}\right)^6+\ldots\right]$$

$$+\frac{e^2}{r}\left[\left\{\left(\frac{a}{r}\right)^n 2\cdot\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}n\cos n\theta+\left(\frac{a}{r}\right)^{n+2}2\cdot\frac{1\cdot3\cdot5\ldots2n+3}{2\cdot4\cdot6\ldots2n+4}\right.\right.$$

$$\cdot\frac{1\cdot n+2}{1\cdot2n+3}n\cos n\theta+\left(\frac{a}{r}\right)^{n+4}2\cdot\frac{1\cdot3\cdot5\ldots2n+7}{2\cdot4\cdot6\ldots2n+8}\cdot\frac{1\cdot3(n+4)(n+3)}{1\cdot2(2n+7)(2n+5)}$$

$$\left.n\cos n\theta+\text{etc.}\right\}+\left\{\left(\frac{a}{r}\right)^{2n}2\cdot\frac{1\cdot3\cdot5\ldots4n-1}{2\cdot4\cdot6\ldots4n}n\cos 2n\theta\right.$$

$$+\left(\frac{a}{r}\right)^{2n+2} 2\cdot\frac{1\cdot3\cdot5\ldots4n+3}{2\cdot4\cdot6\ldots4n+4}\cdot\frac{1\cdot2n+2}{1\cdot4n+3}\,n\cos2n\theta+\left(\frac{a}{r}\right)^{2n+4}\cdot$$

$$2\cdot\frac{1\cdot3\cdot5\ldots4n+7}{2\cdot4\cdot6\ldots4n+8}\cdot\frac{1\cdot3}{1\cdot2}\,\frac{(2n+4)\,(2n+3)}{(4n+7)\,(4n+5)}n\cos2n\theta+\text{etc.}\Big\} + \text{etc.}\ \Big].$$

$$=-\frac{Ze^2}{r}+\frac{ne^2}{r}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{a}{r}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{a}{r}\right)^4+\ldots\right]$$

$$+\frac{2ne^2}{r}\left[f_n(r)\cos n\theta+f_{2n}(r)\cos2n\theta+\text{etc.}\right],$$

where

$$f_n(r)=\alpha_n\left(\frac{a}{r}\right)^{n}+\beta_n\left(\frac{a}{r}\right)^{n+2}+\gamma_n\left(\frac{a}{r}\right)^{n+4}+\ldots,$$

$$f_{2n}(r)=\alpha_{2n}\left(\frac{a}{r}\right)^{2n}+\beta_{2n}\left(\frac{a}{r}\right)^{2n+2}+\gamma_{2n}\left(\frac{a}{r}\right)^{2n+4}+\text{etc.}\ldots,$$

and so on ;

$$\alpha_n=\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}\ ;\ \beta_n=\frac{1}{2}\cdot\frac{2n+1}{2n+2}\alpha_n\ ;$$

$$\gamma_n=\frac{1\cdot3}{8}\cdot\frac{2n+1}{2n+2}\cdot\frac{2n+3}{2n+4}\alpha_n=\frac{3}{4}\cdot\frac{2n+3}{2n+4}\beta_n\ ;\ \text{etc.}$$

$f_{2n}(r)$, $f_{3n}(r)$, etc., being the same functions of r as $f_n(r)$ obtained by simply substituting $2n$, $3n$, etc., instead of n in $f_n(r)$.

(b) *By the Method of Fourier Series.*

As in (a),

$$V=-\frac{Ze^2}{r}+\frac{e^2}{r}\Big[(1-2\mu\cos\theta+\mu^2)^{-\frac{1}{2}}+(1-2\mu\cos\overline{\theta+a}+\mu^2)^{-\frac{1}{2}}$$

$$+(1-2\mu\cos(\theta+2a)+\mu^2)^{-\frac{1}{2}}+\ldots$$

$$+\{1-2\mu\cos(\theta+\overline{n-1}a)+\mu^2\}^{-\frac{1}{2}}\Big]$$

when $\mu=\dfrac{a}{r}<1.$

Hence

$$V=-\frac{Ze^2}{r}+\frac{e^2}{r}\left[\frac{1}{2}\sum_{-\infty}^{\infty}b^{(i)}\cos i\theta+\frac{1}{2}\sum_{-\infty}^{\infty}b^{(i)}\cos i(\theta+a)\right.$$

$$\left.+\frac{1}{2}\sum_{-\infty}^{\infty}b^{(i)}\cos i(\theta+2a)+\ldots+\frac{1}{2}\sum_{-\infty}^{\infty}b^{(i)}\cos i(\theta+\overline{n-1}a)\right],$$

where

$$\frac{1}{2}b^{(i)}=\frac{1\cdot3\cdot5\ldots2i-1}{2\cdot4\cdot6\ldots2i}\mu^i[1+\frac{1}{2}\cdot\frac{2i+1}{2i+2}\mu^2+\frac{1\cdot3}{2\cdot4}\cdot\frac{(2i+1)(2i+3)}{(2i+2)(2i+4)}\cdot$$

$$\mu^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{(2i+1)(2i+3)(2i+5)}{(2i+2)(2i+4)(2i+6)}\mu^6+\ldots];$$

$$\tfrac{1}{2}b^{(0)}=1+\frac{(\frac{1}{2})^2\mu^2}{(1)^2}+\frac{(\frac{1}{2})^2(\frac{1}{2}+1)^2\mu^4}{(1\cdot2)^2}+\ldots+\frac{(\frac{1}{2})^2(\frac{1}{2}+1)^2\ldots(\frac{1}{2}+i-1)^2}{(1\cdot2\cdot3\ldots i)^2}\mu^{2i}$$

$$+\text{etc.},$$

$$=1+(\tfrac{1}{2})^2\mu^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\mu^4+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\mu^6+\ldots$$

$$+\left(\frac{1\cdot3\cdot5\ldots2i-1}{2\cdot4\cdot6\ldots2i}\right)^2\mu^{2i}+\text{etc.};$$

and $$b^{(i)}=b^{(-i)}.$$

[See for instance Tisserand '*Mec. Celeste.*' I. pp. 270-72].

$$\therefore V=-\frac{Ze^2}{r}+\frac{e^2}{r}[\{(\tfrac{1}{2})^2\mu^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\mu^4+\ldots\}\times n+\sum_{-\infty}^{\infty}\frac{1\cdot3\cdot5\cdot2i-1}{2\cdot4\cdot6\ldots2i}\mu^i\cdot$$

$$\{1+\frac{1}{2}\cdot\frac{2i+1}{2i+2}\mu^2+\frac{1\cdot3}{2\cdot4}\,\frac{(2i+1)(2i+3)}{(2i+2)(2i+4)}\mu^4\ldots\ldots\}\{\cos i\theta+\cos i(\theta+a)$$

$$+\ldots+\cos i(\theta+\overline{n-1}a)\}].$$

$$=-\frac{Ze^2}{r}+\frac{ne^2}{r}[1+\left(\frac{1}{2}\right)^2\mu^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\mu^4+\ldots]$$

$$+\sum_{-\infty}^{\infty}\frac{1\cdot3\cdot5\ldots2i-1}{2\cdot4\cdot6\ldots2i}\mu^i\{1+\frac{1}{2}\cdot\frac{2i+1}{2i+2}\mu^2+\frac{1\cdot3}{2\cdot4}\cdot\frac{(2i+1)(2i+3)}{(2i+2)(2i+4)}\mu^4$$

$$+\ldots\}\cos(i\theta+\frac{i}{n}(n-1)\pi)\cdot\frac{\sin i\pi}{\sin\frac{i\pi}{n}}\Bigg],\quad\text{putting in } a=\frac{2\pi}{n}.$$

If $i \neq n$, or $2n$ or $3n$, etc.,

$$\sin i\pi / \sin\frac{i\pi}{n} = 0.$$

If $i = n$, $2n$, etc.,

$$\sin i\pi / \sin\frac{i\pi}{n} = \pm n,$$

according as n is odd or even. Hence

$$\cos\{i\theta + \frac{i}{n}(n-1)\pi\} \frac{\sin i\pi}{\sin\frac{i\pi}{n}} = n\cos n\theta,\ n\cos 2n\theta, \text{ etc.},$$

whether n is even or odd.

We find then

$$V = -\frac{Ze^2}{r} + \frac{ne^2}{r}\left[1 + \left(\frac{1}{2}\right)^2\mu^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\mu^4 + \ldots\right]$$

$$+ \frac{2e^2}{r}\cdot n\left[\frac{1\cdot 3\cdot 5\ldots 2n-1}{2\cdot 4\cdot 6\ldots 2n}\mu^n\left(1 + \frac{1}{2}\cdot\frac{2n+1}{2n+2}\mu^2\right.\right.$$

$$\left. + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(2n+1)(2n+3)}{(2n+2)(2n+4)}\mu^4 + \ldots\right)\cos n\theta$$

$$+ \frac{1\cdot 3\cdot 5\ldots 4n-1}{2\cdot 4\cdot 6\ldots 4n}\mu^{2n}\left(1 + \frac{1}{2}\cdot\frac{4n+1}{4n+2}\mu^2\right.$$

$$\left.\left. + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(4n+1)(4n+3)}{(4n+2)(4n+4)}\mu^4 + \ldots\right)\cos 2n\theta + \ldots\right].$$

$$= -\frac{Ze^2}{r} + \frac{ne^2}{r}\left[1 + \left(\frac{1}{2}\right)^2\mu^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\mu^4 + \ldots\right]$$

$$+ \frac{2ne^2}{r}\left[f_n(r)\cos n\theta + f_{2n}(r)\cos 2n\theta + \ldots\right],$$

where

$$f_n(r) = \frac{1\cdot 3\cdot 5\ldots 2n-1}{2\cdot 4\cdot 6\ldots 2n}\mu^n\left[1 + \frac{1}{2}\cdot\frac{2n+1}{2n+2}\mu^2 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(2n+1)(2n+3)}{(2n+2)(2n+4)}\mu^4 + \ldots\right]$$

$$= \mu^n\alpha^n + \mu^{n+2}\beta_n + \mu^{n+4}\gamma_n + \ldots,$$

if $$a_n = \frac{1\cdot 3\cdot 5\ldots 2n-1}{2\cdot 4\cdot 6\ldots 2n}\ : \qquad \beta_n = \frac{1}{2}\cdot\frac{2n+1}{2n+2}\,a_n\ ;$$

$$\gamma_n = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(2n+1)(2n+3)}{(2n+2)(2n+4)}a_n = \frac{3}{4}\cdot\frac{2n+3}{2n+4}\beta_n\ ;\ \text{etc.}$$

as found out in (a). Similar values of $f_{2n}(r), f_{3n}(r)$ etc.

The expression for this potential may be compared with that given by Sommerfeld.[1] This is

$$V = -\frac{Ze^2}{r} + \frac{ne^2}{r}\left[1 + \left(\frac{1}{2}\right)^2\left(\frac{a}{r}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{a}{r}\right)^4 + \ldots\right].$$

Thus Sommerfeld has got only the first term which is free from θ. The remaining terms involve both r and θ and cannot be neglected except in the exceptional circumstances, except when $\left(\frac{a}{r}\right)^2$ is very small, or r is large compared to a, *i.e.*, for distant orbits like $2p$ or $3d^2$ or 2_2, 3_3 orbits in Bohr's[3] newer notation. For the treatment of (ms) orbits, such approximations are not permissible. This is not allowable in atoms of smaller atomic weight like Lithium where n is small (2 in the case of Lithium), or where in the outer ring, there are more than one electron, *e.g.*, in the case of the alkaline earths or elements of higher groups.

III

EQUATIONS OF MOTION OF THE VALENCY ELECTRON.

Suppose the n-electrons distributed at equal distances on the ring are all describing the same unperturbed circle of radius a with angular velocity ω (there being no mutual perturbations between them) ; and also that the valency electron describes a perturbed circle of mean radius b with normal angular velocity ω'. We define

$$\begin{aligned}\phi &= \omega t + \epsilon,\\ \chi &= \omega' t + \epsilon' + \sigma,\\ &= \theta + \phi,\\ OP = r &= b + \rho,\end{aligned}$$

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[1] '*Atombau und Spektrallinien*,' 2nd ed., *Zusätze und Ergänzungen* 10, p. 507.

[2] Ibid, Chap., VI.

[3] Bohr—loc. cit., p. 20.

ϕ=angle which the radius vector to any one of the inner electrons makes with a line (OL) fixed in space,

θ=angle between the radiivectores to the outer electron at P and the inner electron above referred to.

χ=angle made by OP with the fixed line in space,

ρ is small in comparison to a, b.

σ is always a small angle.

Whence

$$\begin{aligned}\theta &= \chi-(\omega t+\epsilon),\\ &=(\omega'-\omega)t+(\epsilon'-\epsilon)+\sigma,\\ &=\mathrm{I}+\sigma, \text{ say.}\end{aligned}$$

Equations of motion are :

$$\left.\begin{aligned} m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\chi}{dt}\right)^2\right] &= -\frac{\partial V}{\partial r},\\ m\cdot\left[\frac{1}{r}\,\frac{d}{dt}\left(r^2\frac{d\chi}{dt}\right)\right. &= -\frac{1}{r}\,\frac{\partial V}{\partial \chi},\end{aligned}\right\}$$

or

$$\left.\begin{aligned} m\left[\frac{d^2\rho}{dt^2}-2b\omega'\frac{d\sigma}{dt}-b\omega'^2-\rho\omega'^2\right] &= -\frac{\partial V}{\partial r}\\ m\left[b^2\,\frac{d^2\sigma}{dt^2}+2b\omega'\frac{d\rho}{dt}\right] &= -\frac{\partial V}{\partial \chi}.\end{aligned}\right\}$$

Remembering

$$V=-\frac{Ze^2}{r}+\frac{ne^2}{r}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{a}{r}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{a}{r}\right)^4+\dots\right]$$
$$+\frac{2ne^2}{r}\left[f_n(r)\cos n\theta+f_{2n}(r)\cos 2n\theta+\dots\right],$$

we find for a electrically neutral atom, $Z=n+1$, and

$$-\frac{\partial V}{\partial r}=-\frac{e^2}{r^2}+\frac{ne^2}{r^2}\left[\left(\frac{1}{2}\right)^2 a^2\cdot\frac{3}{r^2}+\left(\frac{1\cdot3}{2\cdot4}\right)^2 a^4\cdot\frac{5}{r^4}+\dots\right]$$
$$-2ne^2\left[\cos\theta\cdot\frac{d}{dr}\cdot r^{-2}f_n(r)+\cos 2n\theta\cdot\frac{d}{dr}\cdot r^{-2}f_{2n}(r)+\dots\right].$$

Now

$$\frac{d}{dr}\cdot r^{-1}f_n(r) = -\frac{1}{r^2}\{(n+1)\alpha_n\left(\frac{a}{r}\right)^{n}+(n+3)\beta_n\left(\frac{a}{r}\right)^{n+2}+\dots\},$$

$$\frac{d}{dr}\cdot r^{-1}f_{2n}(r) = -\frac{1}{r^2}\{(2n+1)\alpha_{2n}\left(\frac{a}{r}\right)^{2n}+(2n+3)\beta_{2n}\left(\frac{a}{r}\right)^{2n+2}+\dots\};$$

etc.

Substituting we find

$$-\frac{\partial V}{\partial r} = -\frac{e^2}{r^2}+\frac{ne^2}{r^2}\left[3\cdot\left(\frac{1}{2}\right)^2\cdot\left(\frac{a}{r}\right)^2+5\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{a}{r}\right)^4+\dots\right]$$

$$+\frac{2ne^2}{r^2}\left[\{(n+1)\alpha_n\left(\frac{a}{r}\right)^{n}+(n+3)\beta_n\left(\frac{a}{r}\right)^{n+2}+\dots\}\cos n\theta\right.$$

$$\left.+\{(2n+1)\alpha^{2n}\left(\frac{a}{r}\right)^{2n}+(2n+3)\beta_{2n}\left(\frac{a}{r}\right)^{2n+2}+\dots\}\cos 2n\theta+\dots\right],$$

$$=-\frac{e^2}{b^2}\left(1+\frac{\rho}{b}\right)^{-2}+ne^2\left[3\cdot\left(\frac{1}{2}\right)^2\frac{a^2}{b^4}\left(1+\frac{\rho}{b}\right)^{-4}\right.$$

$$\left.+5\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{a^4}{b^6}\left(1+\frac{\rho}{b}\right)^{-6}+\dots\right]$$

$$+2ne^2\left[\{(n+1)\alpha_n\frac{a^n}{b^{n+2}}\left(1+\frac{\rho}{b}\right)^{-n-2}\right.$$

$$+(n+3)\beta_n\frac{a^{n+2}}{b^{n+4}}\left(1+\frac{\rho}{b}\right)^{-n-4}+\dots\}\cos n\theta$$

$$+\{(2n+1)\alpha_{2n}\frac{a^{2n}}{b^{2n+2}}\left(1+\frac{\rho}{b}\right)^{-2n-2}$$

$$\left.+(2n+3)\beta_{2n}\frac{a^{2n+2}}{b^{2n+4}}\cdot\left(1+\frac{\rho}{b}\right)^{-2n-4}+\dots\}\cos 2n\theta+\text{etc.}\right].$$

Now $\theta=\mathrm{I}+\sigma$; $\cos n\theta=\cos n\mathrm{I}+n\sigma\sin n\mathrm{I}$; etc. Substituting in the above the values of $\cos n\theta$, $\cos 2n\theta$, ..., we find, if $k=a/b$:—

$$-\frac{\partial \mathrm{V}}{\partial r}=-\frac{e^2}{b^2}[1-n\{3\left(\frac{1}{2}\right)^2k^2+5\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\ldots\}]$$

$$+\frac{2ne^2}{b^2}[\{(n+1)\alpha_n k^n+(n+3)\beta_n k^{n+2}+\ldots\}\cos n\mathrm{I}$$

$$+\{(2n+1)\alpha_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\ldots\}\cos 2n\mathrm{I}+\ldots]$$

$$+\frac{e^2\rho}{b^3}[2-n\{3\cdot 4\left(\frac{1}{2}\right)^2k^2+6\cdot 5\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\cdots\}$$

$$-2n\{\left((n+2)(n+1)\alpha_n k^n+(n+4)(n+3)\beta_n k^{n+2}+\ldots\right)\cos n\mathrm{I}$$

$$+\left((2n+2,(2n+1)\alpha_{2n}k^{2n}+(2n+4)(2n+3)\beta_{2n}k^{2n+2}+\ldots\right)\cos 2n\mathrm{I}$$

$$+\ldots\}]+\frac{2n^2e^2}{b^2}\sigma[\{(n+1)\alpha_n k^n+(n+3)\beta_n k^{n+2}+\ldots\}\sin n\mathrm{I}$$

$$+2\{(2n+1)\alpha_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\ldots\}\sin 2n\mathrm{I}+\ldots].$$

$$-\frac{\partial \mathrm{V}}{\partial \chi}=-\frac{2ne^2}{r}[f_n(r)\cdot\frac{\partial}{\partial \chi}\cos n(\chi-\phi)+f_{2n}(r)\cdot\frac{\partial}{\partial \chi}\cos 2n(\chi-\phi)+\ldots]$$

$$=\frac{2n^2e^2}{r}[f_n(r)\sin n(\chi-\phi)+2f_{2n}(r)\sin 2n(\chi-\phi)+\ldots]$$

$$=\frac{2n^2e^2}{r}[f_n(r)\sin n(\mathrm{I}+\sigma)+2f_{2n}(r)\sin 2n(1+\sigma)+\ldots].$$

Now

$$\sin n(1+\sigma)=\sin n\mathrm{I}+n\sigma\cos n\mathrm{I},\ \text{etc.}$$

Substituting we find as before

$$-\frac{\partial \mathrm{V}}{\partial \chi}=\frac{2n^2e^2}{b}[(\alpha_n k^n+\beta_n k^{n+2}+\ldots)\sin n\mathrm{I}$$

$$+2(\alpha_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\ldots)\sin 2n\mathrm{I}+\ldots]$$

$$+\frac{2n^3e^2}{b}\,\sigma[(a_nk^n+\beta_nk^{n+2}+\ldots)\cos n\mathrm{I}$$

$$+2^2\cdot(a_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\ldots)\cos 2n\mathrm{I}+\ldots]$$

$$-\frac{2n^2e^2}{b^2}\,\rho[\{(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\ldots\}\sin n\mathrm{I}$$

$$+2\{(2n+1)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\ldots\}\sin 2n\mathrm{I}+\ldots].$$

The equations of motion can be written thus

$$\frac{d^2\rho}{dt^2}-2b\omega'\frac{d\sigma}{dt}-b\omega'^2-\rho\omega'^2$$

$$=-\frac{e^2}{mb^2}\left[1-n\left\{3\cdot\left(\frac{1}{2}\right)^2k^2+5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\ldots\right\}\right]$$

$$+\frac{2ne^2}{b^2m}[\{(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\ldots\}\cos n\mathrm{I}+\ldots]$$

$$+\frac{e^2\rho}{b^3m}\left[2-n\left\{3\cdot4\left(\frac{1}{2}\right)^2k^2-6\cdot5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\ldots\right\}\right.$$

$$\left.-2n\{(n+2)(n+1)a_nk^n+(n+4)(n+3)\beta_nk^{n+2}+\ldots)\cos n\mathrm{I}+\ldots\}\right]$$

$$+\frac{2n^2e^2}{b^2m}\,\sigma[\{(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\ldots\}\sin n\mathrm{I}$$

$$+2\{(2n+1)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\ldots\}\sin 2n\mathrm{I}+\ldots]\quad\ldots\quad\mathrm{I.}$$

$$b\,\frac{d^2\sigma}{dt^2}+2\omega'\frac{d\rho}{dt}=\frac{2n^2e^2}{b^2m}\,[(a_nk^n+\beta_nk^{n+2}+\cdots)\sin n\mathrm{I}$$

$$+2(a_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\cdots)\sin 2n\mathrm{I}+\cdots]+\frac{2n^3e^2}{b^3m}\,\sigma[(a_nk^n$$

$$+\beta_nk^{n+2}+\cdots)\cos n\mathrm{I}+2^2\cdot(a_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\cdots)\cos 2n\mathrm{I}+\ldots]$$

$$-\frac{2n^2e^2}{b^3m}\,\rho[\{(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\cdots\}\sin n\mathrm{I}$$

$$+2\{(2n+1)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots\}\sin 2n\mathrm{I}+\cdots]\quad\ldots\quad\mathrm{II.}$$

Now $\qquad \mathrm{I}=(\omega'-\omega)t+(\epsilon'-\epsilon)$

$$\therefore \frac{d}{dt}=(\omega'-\omega)\frac{d}{d\mathrm{I}}\,; \qquad \frac{d^2}{dt^2}=(\omega'-\omega)^2\frac{d^2}{d\mathrm{I}^2}\,.$$

Equations I, II can then be written in the form

$$\frac{d^2\rho}{d\mathrm{I}^2}-2b\frac{\omega'}{\omega'-\omega}\;\frac{d\sigma}{d\mathrm{I}}-b\frac{\omega'^2}{(\omega'-\omega)^2}$$

$$-\rho\frac{\omega'^2}{(\omega'-\omega)^2}=\frac{1}{(\omega'-\omega)^2}\ [\textit{idem}\text{ of Equation I}] \quad \ldots \quad \mathrm{I}'.$$

$$\frac{d^2\sigma}{d\mathrm{I}^2}+2\frac{\omega'}{\omega'-\omega}\;\frac{d\rho}{d\mathrm{I}}=\frac{1}{b(\omega'-\omega)^2}\ [\textit{idem}\text{ of Equation II}] \quad \ldots \quad \mathrm{II}'$$

For the sake of homogeneity write $b\rho$ for ρ, so that

$$r=b(1+\rho).$$

Hence, we have

$$\frac{d^2\rho}{d\mathrm{I}^2}-2\frac{\omega'}{\omega'-\omega}\;\frac{d\sigma}{d\mathrm{I}}-\frac{\omega'^2}{(\omega'-\omega)^2}-\rho\frac{\omega'^2}{(\omega'-\omega)^2}=\frac{1}{b(\omega'-\omega)^2}\ [\textit{idem}] \quad \ldots \quad \mathrm{I}''$$

$$\frac{d^2\sigma}{d\mathrm{I}^2}+2\frac{\omega'}{\omega'-\omega}\;\frac{d\rho}{d\mathrm{I}}=\frac{1}{b(\omega'-\omega)^2}\ [\textit{idem}] \quad \ldots \quad \mathrm{II}''$$

Now put $-\omega'/\omega—\omega'=\nu$, so that

$$\frac{1}{b^3(\omega-\omega')^2}=\frac{\nu^2}{b^3\omega'^2}=\frac{\nu^2}{e^2/m}\,.$$

Because

$$mb\omega'^2=\frac{\partial\mathrm{V}}{\partial r}=\frac{e^2}{b^2}$$

approximately, as can be seen from the value of $-\dfrac{\partial\mathrm{V}}{\partial r}$ found above.

If $(\rho'', \sigma'', \rho', \sigma')$ stand for

$$\left(\frac{d^2\rho}{d\mathrm{I}^2},\ \frac{d^2\sigma}{d\mathrm{I}^2},\ \frac{d\rho}{d\mathrm{I}},\ \frac{d\sigma}{d\mathrm{I}}\right)$$

respectively, the equations I″, II″ can be re-written in the form thus

$$\rho''-2\nu\sigma'-\nu^2-\rho\nu^2=-\nu^2\left[1-n\left\{3\cdot\left(\frac{1}{2}\right)^2k^2+5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdots\right\}\right]$$

$$+2n\nu^2\left[\left\{(n+1)\alpha_n k^n+(n+3)\beta_n k^{n+2}+\cdots\right\}\cos n\mathrm{I}\right.$$

$$\left.+\left\{(2n+1)\alpha_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots\right\}\cos 2n\mathrm{I}+\cdots\right]$$

$$+\nu^2\rho\left[2-n\left\{3\cdot4\cdot\left(\frac{1}{2}\right)^2k^2+6\cdot5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdots\right\}\right.$$

$$-2n\left\{\left((n+2)(n+1)\alpha_n k^n+(n+4)(n+3)\beta_n k^{n+2}+\cdots\right)\cos n\mathrm{I}\right.$$

$$+\left((2n+2)(2n+1)\alpha_{2n}k^{2n}+(2n+4)(2n+3)\beta_{2n}k^{2n+2}+\cdots\right)\cos 2n\mathrm{I}$$

$$\left.\left.+\cdots\right\}\right]+2n^2\nu^2\sigma\left[\left\{(n+1)\alpha_n k^n+(n+3)\beta_n k^{n+2}+\cdots\right\}\sin n\mathrm{I}\right.$$

$$\left.+2\left\{(2n+1)\alpha_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots\right\}\sin 2n\mathrm{I}+\cdots\right] \quad \ldots \quad (3)$$

$$\sigma''+2\nu\rho'=2n^2\nu^2\left[(\alpha_n k^n+\beta_n k^{n+2}+\cdots)\sin n\mathrm{I}+2(\alpha_{2n}k^{2n}+\beta_{2n}k^{2n+2}\right.$$

$$\left.+\cdots)\sin 2n\mathrm{I}+\cdots\right]+2n^3\nu^4\sigma\left[(\alpha_n k^n+\beta_n k^{n+2}+\cdots)\cos n\mathrm{I}\right.$$

$$\left.2^2(\alpha_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\cdots)\cos 2n\mathrm{I}+\cdots\right]$$

$$-2n^2\nu^2\rho\left[\left\{(n+1)\alpha_n k^n+(n+3)\beta_n k^{n+2}+\cdots\right\}\sin n\mathrm{I}\right.$$

$$\left.+2\left\{(2n+1)\alpha_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots\right\}\sin 2n\mathrm{I}+\cdots\right] \ldots \quad (4)$$

Equations (3) and (4) can be briefly expressed thus :

$$\rho''-2\nu\sigma'+(\odot_{1,0}+\odot_{1,1}\cos n\mathrm{I}+\odot_{1,2}\cos 2n\mathrm{I}+\cdots)\rho$$

$$+(\odot_{2,1}\sin n\mathrm{I}+\odot_{2,2}\sin 2n\mathrm{I}+\cdots)\sigma$$

$$=\odot_{3,0}+\odot_{3,1}\cos n\mathrm{I}+\odot_{3,2}\cos 2n\mathrm{I}+\cdots \quad \ldots \quad (5)$$

$$\sigma''+2\nu\rho'+(\odot_{4,1}\sin n\mathrm{I}+\odot_{4,2}\sin 2n\mathrm{I}+\cdots)\rho$$

$$+(\odot_{5,1}\cos n\mathrm{I}+\odot_{5,2}\cos 2n\mathrm{I}+\cdots)\sigma$$

$$=\odot_{6,1}\sin n\mathrm{I}+\odot_{6,2}\sin 2n\mathrm{I}+\cdots \quad \ldots \quad (6)$$

The values of $\odot$'s are given below :

$$\odot_{1,0}=\nu^2[-3+n\{3\cdot4\cdot\left(\frac{1}{2}\right)^2k^2+6\cdot5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdot\cdot] ;$$

$$\odot_{1,1}=2n\nu^2[(n+2)(n+1)a_nk^n+(n+4)(n+3(\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{1,2}=2n\nu^2[(2n+2)(2n+1)a_{2n}k^{2n}+(2n+4)(2n+3)\beta_{2n}k^{2n+2}+\cdots] ;$$

$$\odot_{2,1}=-2n\nu^2[(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{2,2}=-4n^2\nu^2[(2n+1)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots] ;$$

$$\odot_{3,0}=n\nu^2[3\cdot\left(\frac{1}{2}\right)^2k^2+5\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdots] ;$$

$$\odot_{3,1}=2n\nu^2[(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{3,2}=2n\nu^2[(2n+2)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots] ;$$

$$\odot_{4,1}=2n^2\nu^2[(n+1)a_nk^n+(n+3)\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{4,2}=4n^2\nu^2[(2n+1)a_{2n}k^{2n}+(2n+3)\beta_{2n}k^{2n+2}+\cdots] ;$$

$$\odot_{5,1}=-2n^3\nu^2[a_nk^n+\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{5,2}=-8n^3\nu^2[a_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\cdots] ;$$

$$\odot_{6,1}=2n^2\nu^2[a_nk^n+\beta_nk^{n+2}+\cdots] ;$$

$$\odot_{6,2}=4n^2\nu^2[a_{2n}k^{2n}+\beta_{2n}k^{2n+2}+\cdot\cdot\ ] ;$$

and so on. Since ν, k are fractional we see that among the $\odot$'s, only $\odot_{1,0}$, $\odot_{3,0}$ are larger than others (provided $n>2$) by an appreciable amount.

IV

SOLUTION OF THE EQUATIONS.

(a) *The Complementary Function*

The equations

$$\left.\begin{aligned}\rho''-2\nu\sigma'+\rho\sum_{r=0}^{\infty}\odot_{1,r}\cos rn\mathrm{I}+\sigma\sum_{r=1}^{\infty}\odot_{2,r}\sin rn\mathrm{I}=0\\ \sigma''+2\nu\rho'+\rho\sum_{r=1}^{\infty}\odot_{4,r}\sin rn\mathrm{I}+\sigma\sum_{r=1}^{\infty}\odot_{5,r}\cos rn\mathrm{I}=0\end{aligned}\right\}\quad\ldots\quad[\mathrm{A}]$$

are homogeneous linear differential equations with periodic coefficients, and the solutions, as will be seen, are quasi-periodic solutions of the type $e^{cI}\phi(I)$, where $\phi(I)$ is a periodic function having the same period as the coefficients in the above equations and the parameter c is the factor of quasi-periodicity. The crux of the problem centres round a complete determination of this important parameter. Equations of this nature with *one dependent variable* have been discussed by Hill,[1] Young,[2] Ince,[3] Baker[4] and Whittaker;[5] and were first used by Hill in his classical studies on the perturbations of the moon. The present equation is, however, of a more general type as it involves *two dependent variables* instead of *one* as in Hill's equation; a modification of their methods first introduced by Goldsbrough[6] will be employed here for an integral of these equations, as Hill's general analysis involves an evaluation of infinite determinant in c, which is unmanagable.

Suppose

$$\rho = e^{cI} \cdot A,$$

$$\sigma = e^{cI} \cdot X,$$

where A and X are purely periodic functions of period 2π. On substituting in equations (A) we find, since

$$\frac{d^2}{dI^2}(e^{cI} A) = e^{cI} A'' + 2ce^{cI} A' + c^2 e^{cI} A,$$

$$\frac{d^2}{dI^2}(e^{cI} X) = e^{cI} X'' + 2ce^{cI} X' + c^2 e^{cI} X,$$

$$\frac{d}{dI}(e^{cI} A) = e^{cI} A' + ce^{cI} A.$$

$$\frac{d}{dI}(e^{cI} X) = e^{cI} X' + ce^{cI} X,$$

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[1] '*Acta Mathematica,*' Vol. VIII; Whittaker's '*Modern Analysis.*'

[2] '*Proc. Edin. Math. Soc.,*' XXXII, p. 81.

[3] '*M. N. R. A. S.*', LXXV. 5, p. 436.

[4] '*Phil. Trans. A.*', Vol. 216, p. 129.

[5] '*Proc. Inter. Congress. Math.*', Vol. I, 1912; '*Proc. Edin. Math. Soc.*', XXXII, p. 76; '*Modern Analysis.*'

[6] '*Phil Trans.*' Vol. 222, 1922.

$$\left.\begin{aligned} c^2A+2cA'+A''-2\nu(cX+X')+A\sum_{r=0}^{\infty}\odot_{1,r}\cos rnI & \\ +X\sum_{r=1}^{\infty}\odot_{2,r}\sin rnI=0 & \\ c^2X+2cX'+X''+2\nu(cA+A')+A\sum_{r=1}^{\infty}\odot_{4,r}\sin rnI & \\ +X\sum_{r=1}^{\infty}\odot_{5,r}\cos rnI=0 & \end{aligned}\right\} \quad \ldots \quad [B]$$

As a general case of Whittaker's solution (Proc. Edin. Math. Soc., loc. cit. p. 77) of Mathew's differential equation in periodic functions let us assume series in $\odot$'s and multiples of $\odot$'s having coefficients with period 2π, *i.e.*

Let
$$\begin{aligned} A &= A_0\sin(\lambda nI-\tau)+\Sigma\Sigma A_{r,s}\odot_{r,s} \\ &\quad +\Sigma\Sigma\Sigma\Sigma B_{r,s,p,q}\,\odot_{r,s}\odot_{p,q}+\cdots \qquad \ldots \quad [E\cdot 1] \\ X &= X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X_{r,s}\odot_{r,s} \\ &\quad +\Sigma\Sigma\Sigma\Sigma Y_{r,s,p,q}\,\odot_{r,s}\odot_{p,q}+\cdots \qquad \ldots \quad [E\cdot 2] \end{aligned}$$

Here A_0, X_0 are arbitrary constants, λ is an arbitrary *integer* and τ a parameter which will be defined presently. As usual let

$$c=\Sigma\Sigma c_{r,s}\odot_{r,s}+\Sigma\Sigma\Sigma\Sigma d_{r,s,p,q}\,\odot_{r,s}\odot_{p,q}+\cdots,$$

where the coefficients of $\odot$'s and multiples of $\odot$'s are functions of n, λ and τ.

Substitute these values of A, X and c in equations [B]. Thus

$$\begin{aligned} &\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}^2\{A_0\sin(\lambda nI-\tau)+\Sigma\Sigma A_{r,s}\odot_{r,s}+\cdots\} \\ &\quad +2\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}\{A_0 n\lambda\cos(\lambda nI-\tau)+\Sigma\Sigma A'_{r,s}\odot_{r,s}+\cdots\} \\ &\quad +\{-A_0\lambda^2n^2\sin(\lambda nI-\tau)+\Sigma\Sigma A''_{r,s}\odot_{r,s}+\cdots\} \\ &\quad -2\nu[\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}\{X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X_{r,s}\odot_{r,s}+\cdots\} \\ &\quad +\{-X_0\lambda n\sin(\lambda nI-\tau)+\Sigma\Sigma X'_{r,s}\odot_{r,s}+\cdots\}] \\ &\quad +\{A_0\sin(\lambda nI-\tau)+\Sigma\Sigma A_{r,s}\odot_{r,s}+\cdots\}\sum_{r=0}^{\infty}\odot_{1,r}\cos rnI \\ &\quad +\{X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X_{r,s}\odot_{r,s}+\cdots\}\sum_{r=1}^{\infty}\odot_{2,r}\sin rnI=0 \quad \ldots \quad [C] \end{aligned}$$

and

$$\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}^2\{X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X_{r,s}\odot_{r,s}+\cdots\}$$
$$+2\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}\{-X_0\lambda n\sin(\lambda nI-\tau)+\Sigma\Sigma X'_{r,s}\odot_{r,s}+\cdots\}$$
$$+\{-\lambda^2n^2X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X''_{r,s}\odot_{r,s}+\cdots\}$$
$$+2\nu[\{\Sigma\Sigma c_{r,s}\odot_{r,s}+\cdots\}\{A_0\sin(\lambda nI-\tau)+\Sigma\Sigma A_{r,s}\odot_{r,s}+\cdots\}$$
$$+\{A_0\lambda n\cos(\lambda nI-\tau)+\Sigma\Sigma A'_{r,s}\odot_{r,s}+\cdots\}]$$
$$+\{A_0\sin(\lambda nI-\tau)+\Sigma\Sigma A_{r,s}\odot_{r,s}+\cdots\}\sum_{r=1}^{\infty}\odot_{1,r}\sin rnI$$
$$+\{X_0\cos(\lambda nI-\tau)+\Sigma\Sigma X_{r,s}\odot_{r,s}+\cdots\}\sum_{r=1}^{\infty}\odot_{1,r}\cos rnI=0 \qquad [D]$$

First equate to zero those terms not involving any $\odot$. Since in the above series we have not included $\odot_{1,0}$ which is large compared with the others, we shall get $\odot_{1,0}$ on equating, in the $\odot$-independent terms.

Thus

$$\{(\odot_{1,0}-\lambda^2n^2)A_0+2\nu\lambda nX_0\}\sin(\lambda nI-\tau)=0$$
$$\{2\nu\lambda nA_0+(0-\lambda^2n^2)X_0\}\cos(\lambda nI-\tau)=0.$$

Now A_0, X_0 being assumed not equal to zero, on eliminating A_0, X_0 we get $\odot_{1,0}=\lambda^2n^2-4\nu^2$. In general the given value of $\odot_{1,0}$ will not satisfy this equation for any integral value of λ. Suppose $a_{1,0}$ is a quantity which satisfies $a_{1,0}=\lambda^2n^2-4\nu^2$, for some integral value of λ, where $a_{1,0}$, of course, slightly differs from $\odot_{1,0}$.

Now assume

$$\odot_{1,0}=a_{1,0}+\Sigma\Sigma a_{r,s}\odot_{r,s}+\Sigma\Sigma\Sigma\Sigma b_{r,s,p,q}\odot_{r,s}\odot_{p,q}+\cdots\ \ldots \qquad [E]$$

To determine all the unknown coefficients we should impose two conditions.

(*i*) The series for A does not contain the term $\cos(\lambda nI-\tau)$: in fact in this really constitutes the definition of the parameter τ and the possibility of obtaining series which remains convergent for all real values of τ depends upon our choosing τ in this way.

(*ii*) The solutions for A and X must be purely periodic with period 2π (*i.e.*, no part of the exponent shall appear in the periodic series).

Further, these conditions will determine uniquely the undetermined coefficients in the series for c and $\odot_{1,0}$.

On substituting the assumed series for A, X, c and $\odot_{1,0}$ in equation [B] and equating to zero the terms involving $\odot_{1,r}$, as from equations [C] and [D] we find

$$2c_{1,r}\lambda n A_0\cos(\lambda n I-\tau)+A''_{1,r}-2\nu c_{1,r}X_0\cos(\lambda n I-\tau)-2\nu X'_{1,r}$$

$$+a_{1,0}A_{1,r}+a_{1,r}A_0\sin(\lambda n I-\tau)+A_0\cos rnI\sin(\lambda n I-\tau)=0 \;\ldots\; [C\cdot 1]$$

$$-2c_{1,r}\lambda n X_0\sin(\lambda n I-\tau)+X''_{1,r}+2\nu c_{1,r}A_0\sin(\lambda n I-\tau)+2\nu A'_{1,r}=0 \;\ldots\; [D\cdot 1]$$

Now

$$\cos rnI\sin(\lambda n I-\tau)=\tfrac{1}{2}[\sin\{n(r+\lambda)I-\tau\}+\sin\{n(\lambda-r)I-\tau\}]$$

When $r\neq\lambda$ or 2λ, it is clear that

$$c_{1,r}=0, \qquad a_{1,r}=0.$$

Equations [C·1], [D·1] reduce to

$$A''_{1,r}-2\nu X'_{1,r}+a_{1,0}A_{1,r}+\tfrac{1}{2}A_0[\sin\{n(\lambda+r)I-\tau\}+\sin\{n(\lambda-r)I-\tau\}]=0,$$

$$X''_{1,r}+2\nu A'_{1,r}=0,$$

Putting

$$A_{1,r}=p\sin\{(\lambda+r)nI-\tau\}+q\sin\{(\lambda-r)nI-\tau\},$$

$$X_{1,r}=p'\cos\{(\lambda+r)nI-\tau\}+q'\cos\{(\lambda-r)nI-\tau\},$$

where (p, q, p', q') are constants and solving as usual we find

$$A_{1,r}=\frac{A_0\sin\{(\lambda+r)nI-\tau\}}{2rn(2\lambda n+rn)}-\frac{A_0\sin\{(\lambda-r)nI-\tau\}}{2rn(2\lambda n-rn)},$$

$$X_{1,r}=\frac{A_0\nu\cos\{(\lambda+r)nI-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)}-\frac{A_0\nu\cos\{(\lambda-r)nI-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)}.$$

In the special case, when $r=\lambda$, we have

$$\left.\begin{aligned}&A''_{1,\lambda}-2\nu X'_{1,\lambda}+a_{1,0}A_{1,\lambda}+\tfrac{1}{2}A_0[\sin(2\lambda nI-\tau)-\sin\tau]=0\\&X''_{1,\lambda}+2\nu A'_{1,\lambda}=0\end{aligned}\right\}\;[F\cdot 1]$$

These give

$$c_{1,\lambda}=0, \qquad a_{1,\lambda}=0$$

$$A_{1,\lambda}=\frac{A_0\sin(2\lambda nI-\tau)}{6\lambda^2 n^2}+\frac{A_0\sin\tau}{a_{1,0}};$$

$$X_{1,\lambda}=\frac{A_0\nu\cos(2\lambda nI-\tau)}{6\lambda^3 n^3}.$$

When $r=2\lambda$, equations [C·1], [D·1] can be written in the form:

$$\begin{aligned}&2c_{1,2\lambda}\lambda nA_0\cos(\lambda nI-\tau)+A''_{1,2\lambda}-2\nu c_{1,2\lambda}X_0\cos(\lambda nI-\tau)\\&\quad-2\nu X'_{1,2\lambda}+a_{1,0}A_{1,2\lambda}+a_{1,2\lambda}A_0\sin(\lambda nI-\tau)+\tfrac{1}{4}A_0[\sin(3\lambda nI-\tau)\\&\quad-\sin(\lambda nI-\tau)\cos2\tau-\cos(\lambda nI-\tau)\sin2\tau]=0 \qquad \ldots \quad [\mathrm{C}\cdot2]\end{aligned}$$

$$\begin{aligned}&-2c_{1,2\lambda}\lambda nX_0\sin(\lambda nI-\tau)+X''_{1,2\lambda}\\&\qquad\qquad+2\nu c_{1,2\lambda}A_0\sin(\lambda nI-\tau)+2\nu A'_{1,2\lambda}=0 \qquad \ldots \quad [\mathrm{D}\cdot2]\end{aligned}$$

To obtain $a_{1,2\lambda}$ we collect the $\sin(\lambda nI-\tau)$ terms, thus

$$\left.\begin{aligned}&A''_{1,2\lambda}-2\nu X'_{1,2\lambda}+a_{1,0}A_{1,2\lambda}\\&\qquad+A_0\sin(\lambda nI-\tau)[a_{1,2\lambda}-\tfrac{1}{4}\cos2\tau]=0\\&\qquad\qquad X''_{1,2\lambda}+2\nu A'_{1,2\lambda}=0\end{aligned}\right\}\qquad[\mathrm{F}\cdot2]$$

whence we get

$$a_{1,2\lambda}=\tfrac{1}{4}\cos2\tau,$$

taking solutions for

$$A_{1,2\lambda},\ X_{1,2\lambda}$$

of the form

$$p\sin(\lambda nI-\tau),\ q\cos(\lambda nI-\tau)$$

respectively and remembering

$$a_{1,0}=\lambda^2n^2-4\nu^2.$$

To obtain $c_{1,2\lambda}$, since A must not contain $\cos(\lambda n\mathrm{I}-\tau)$ term (X may contain), we get from the equations

$$\left.\begin{array}{ll}(2c_{1,2\lambda}\lambda n\mathrm{A}_0-2\nu c_{1,2\lambda}\mathrm{X}_0-\tfrac{1}{4}\mathrm{A}_0\sin 2\tau)\cos(\lambda n\mathrm{I}-\tau)-2\nu\mathrm{X}'_{1,2\lambda} & =0\\ (-2c_{1,2\lambda}\lambda n\mathrm{X}_0+2\nu c_{1,2\lambda}\mathrm{A}_0)\sin(\lambda n\mathrm{I}-\tau)+\mathrm{X}''_{1,2\lambda} & =0\end{array}\right\}[\mathrm{F}\cdot 3]$$

$$c_{1,2\lambda}=\frac{1}{4}\,\frac{\sin 2\tau}{\lambda n},$$

remembering

$$\lambda n\mathrm{X}_0=2\nu\mathrm{A}_0 \text{ and } a_{1,0}=\lambda^2n^2-4\nu^2.$$

Substituting this value of $c_{1,2\lambda}$ in either of the equations [F·3] we find

$$\mathrm{X}_{1,2\lambda}=\frac{-\nu\mathrm{A}_0\sin 2\tau\sin(\lambda n\mathrm{I}-\tau)}{2\lambda^3n^3}.$$

To obtain solutions of $\mathrm{A}_{1,2\lambda}$, $\mathrm{X}_{1,2\lambda}$ in $\sin(3\lambda n\mathrm{I}-\tau)$, $\cos(3\lambda n\mathrm{I}-\tau)$ terms write

$$\left.\begin{array}{ll}\mathrm{A}''_{1,2\lambda}-2\nu\mathrm{X}'_{1,2\lambda}+a_{1,0}\mathrm{A}_{1,2\lambda}+\tfrac{1}{2}\mathrm{A}_0\sin(3\lambda n\mathrm{I}-\tau) & =0\\ \mathrm{X}''_{1,2\lambda}+2\nu\mathrm{A}'_{1,2\lambda} & =0\end{array}\right\}\quad\ldots\quad[\mathrm{F}\cdot 4]$$

Assume

$$\mathrm{A}_{1,2\lambda}=p\sin(3\lambda n\mathrm{I}-\tau),$$
$$\mathrm{X}_{1,2\lambda}=q\cos(3\lambda n\mathrm{I}-\tau).$$

Substituting in [F·4] and solving in p and q we find

$$p=\frac{1}{16}\,\frac{\mathrm{A}_0}{\lambda^2n^2};\quad q=\frac{1}{24}\,\frac{\nu\mathrm{A}_0}{\lambda^3n^3}$$

i.e., the particular solutions arising from the term $\tfrac{1}{2}\mathrm{A}_0\sin(3\lambda n\mathrm{I}-\tau)$ are

$$\mathrm{A}_{1,2\lambda}=\frac{1}{16}\,\frac{\mathrm{A}_0}{\lambda^2n^2}\sin(3\lambda n\mathrm{I}-\tau),$$

$$\mathrm{X}_{1,2\lambda}=\frac{1}{24}\,\frac{\nu\mathrm{A}_0}{\lambda^3n^3}\cos(3\lambda n\mathrm{I}-\tau),$$

or, the complete solution for $X_{1,2\lambda}$ is

$$X_{1,2\lambda}=-\frac{1}{2}\frac{\nu A_0}{\lambda^3 n^3}\sin 2\tau\sin(\lambda nI-\tau)+\frac{1}{24}\frac{\nu A_0}{\lambda^3 n^3}\cos(3\lambda nI-\tau).$$

Proceeding in the same mechanical process we can find out the other coefficients of $\odot$'s in the series for A, X, and the undetermined constants in the series for c and $\odot_{1,0}$. We write down the complete results thus.

Terms involving argument $\odot_{1,r}$:

$$c_{1,r}=0, \qquad a_{1,r}=0,$$

$$A_{1,r}=\frac{A_0\sin\{n(\lambda+r)I-\tau\}}{2rn(2\lambda n+rn)}-\frac{A_0\sin\{n(\lambda-r)I-\tau\}}{2rn(2\lambda n-rn)};$$

$$X_{1,r}=\frac{A_0\nu\cos\{n(\lambda+r)I-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)}-\frac{A_0\nu\cos\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)}.$$

Terms involving argument $\odot_{1,\lambda}$:

$$c_{1,\lambda}=0, \qquad a_{1,\lambda}=0.$$

$$A_{1\lambda}=\frac{A_0\sin(2\lambda nI-\tau)}{6\lambda^2 n^2}+\frac{A_0\sin\tau}{2a_{1,0}};$$

$$X_{1,\lambda}=\frac{A_0\nu\cos(2\lambda nI-\tau)}{6\lambda^3 n^3}.$$

Terms involving argument $\odot_{1,2\lambda}$:

$$c_{1,2\lambda}=\frac{1}{4}\frac{\sin 2\tau}{\lambda n}, \quad a_{1,2\lambda}=\tfrac{1}{4}\cos 2\tau.$$

$$A_{1,2\lambda}=\frac{1}{16}\frac{A_0}{\lambda^2 n^2}\sin(3\lambda nI-\tau);$$

$$X_{1,2\lambda}=-\frac{1}{2}\frac{\nu A_0}{\lambda^3 n^3}\sin 2\tau\sin(\lambda nI-\tau)+\frac{1}{24}\frac{\nu A_0}{\lambda^3 n^3}\cos(3\lambda nI-\tau).$$

Terms involving argument $\odot_{2,r}$:

$$c_{2,r}=0, \qquad a_{2,r}=0.$$

$$A_{2,r}=\frac{X_0\sin\{n(\lambda+r)I-\tau\}}{2rn(2\lambda n+rn)}+\frac{X_0\sin\{n(\lambda-r)I-\tau\}}{2rn(2\lambda n-rn)};$$

$$X_{2,r}=\frac{X_0\nu\cos\{n(\lambda+r)I-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)}+\frac{X_0\nu\cos\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)}.$$

Terms involving argument $\odot_{2,\lambda}$:

$$c_{2,\lambda}=0, \qquad a_{2,\lambda}=0.$$

$$A_{2,\lambda}=\frac{X_0\sin(2\lambda nI-\tau)}{6\lambda^2n^2}-\frac{X_0\sin\tau}{2a_{1,0}};$$

$$X_{2,\lambda}=\frac{X_0\nu\cos(2\lambda nI-\tau)}{6\lambda^3n^3}.$$

Terms involving argument $\odot_{2,\,2\lambda}$:

$$c_{2,\,2\lambda}=-\frac{\nu\sin 2\tau}{2\lambda^2n^2}, \qquad a_{2,\,2\lambda}=\frac{-\nu\cos 2\tau}{\lambda n}.$$

$$A_{2,\,2\lambda}=\frac{1}{16}\,\frac{X_0\sin(3\lambda nI-\tau)}{\lambda^2n^2};$$

$$X_{2,\,2\lambda}=\frac{X_0\nu\sin 2\tau\sin(\lambda nI-\tau)}{2\lambda^3n^3}+\frac{1}{24}\,\frac{\nu X_0\cos(3\lambda nI-\tau)}{\lambda^3n^3}.$$

Terms involving argument $\odot_{4,r}$:

$$c_{4,r}=0, \qquad a_{4,r}=0.$$

$$A_{4,r}=\frac{-\nu A_0\sin\{n(\lambda+r)I-\tau)}{rn(\lambda n+rn)(2\lambda n+rn)}-\frac{\nu A_0\sin\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)};$$

$$X_{4,r}=\frac{A_0\{a_{1,0}-(\lambda n+rn)^2\}\cos\{n(\lambda+r)I-\tau\}}{2rn(\lambda n+rn)^2(2\lambda n+rn)}$$

$$+\frac{A_0\{a_{1,0}-(\lambda n-rn)^2\}\cos\{n(\lambda-r)I-\tau\}}{2rn(\lambda n-rn)^2(2\lambda n-rn)};$$

Terms involving argument $\odot_{4,\lambda}$:

$$c_{4,\lambda}=0, \qquad a_{4,\lambda}=0.$$

$$A_{4,\lambda}=\frac{-\nu A_0\sin(2\lambda nI-\tau)}{6\lambda^3 n^3}\ ;$$

$$X_{4,r}=\frac{A_0(a_{1,0}-4\lambda^2n^2)\cos(2\lambda nI-\tau)}{24\lambda^4n^4}.$$

[here in addition $\cos\tau=0$, in order to avoid the existence of I occurring explicitly in $A_{4,\lambda}$.]

Terms involving argument $\odot_{4,\,2\lambda}$:

$$c_{4,\,2\lambda}=\frac{-\nu\sin 2\tau}{2\lambda^2n^2}, \qquad a_{4,\,2\lambda}=-\frac{\nu\cos 2\tau}{\lambda n}.$$

$$A_{4,\,2\lambda}=-\frac{1}{24}\ \frac{A_0\nu\sin(3\lambda nI-\tau)}{\lambda^3n^3}\ ;$$

$$X_{4,\,2\lambda}=\frac{1}{2}\ \frac{A_0\cos 2\tau\cos(\lambda nI-\tau)}{\lambda^2n^2}-\frac{A_0\sin 2\tau(a_{1,0}+2\nu^2)}{2\lambda^3n^3}$$

$$+\frac{1}{16}\ \frac{A_0(\nu^2+2\lambda^2n^2)}{\lambda^4n^4}\ \cos(3\lambda nI-\tau).$$

Terms involving argument $\odot_{5,r}$:

$$c_{5,r}=0, \qquad a_{5,r}=0.$$

$$A_{5,r}=\frac{X_0\nu\sin\{n(\lambda+r)I-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)}-\frac{X_0\nu\sin\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)}\ ;$$

$$X_{5,r}=-\frac{X_0\{a_{1,0}-(\lambda n+rn)^2\}\cos\{n(\lambda+r)I-\tau\}}{2rn(\lambda n+rn)^2(2\lambda n+rn)}$$

$$+\frac{X_0\{a_{1,0}-(\lambda n-rn)^2\}\cos\{n(\lambda-r)I-\tau\}}{2rn(\lambda n-rn)^2(2\lambda n-rn)}.$$

Terms involving argument $\odot_{5,\lambda}$:

$$c_{5,\lambda}=0, \qquad a_{5,\lambda}=0.$$

$$A_{5,\lambda}=\frac{1}{6}\frac{X_0\nu\sin(2\lambda n\mathrm{I}-\tau)}{\lambda^3 n^3};$$

$$X_{5,\lambda}=\frac{-X_0(a_{1,0}-4\lambda^2n^2)\cos(2\lambda n\mathrm{I}-\tau)}{24\lambda^4n^4}$$

[here in addition $\cos\tau=0$, in order to avoid the existence of I occurring explicitly in $A_{5,\lambda}$].

Terms involving argument $\odot_{5,\,2\lambda}$:

$$c_{5,\,2\lambda}=\frac{-\nu\sin2\tau}{2\lambda^2n^2}; \qquad a_{5,\,2\lambda}=\frac{-2\nu^2\cos2\tau}{\lambda^2n^2}.$$

$$A_{5,\,2\lambda}=\frac{X_0\nu\sin(3\lambda n\mathrm{I}-\tau)}{24\lambda^3n^3};$$

$$X_{5,\,2\lambda}=\frac{X_0(2\lambda^2n^2+\nu^2)\cos(3\lambda n\mathrm{I}-\tau)}{36\lambda^4n^4}+\frac{X_0\cos2\tau\cos(\lambda n\mathrm{I}-\tau)}{2\lambda^2n^2}$$

$$+\frac{X_0(\lambda^2n^2-2\nu^2)\sin2\tau\sin(\lambda n\mathrm{I}-\tau)}{4\lambda^3n^3}.$$

Terms involving products powers of $\odot$'s follow in a similar fashion.

If we summarise the parts specially required we find

$$\odot_{1,0}=(\lambda^2n^2-4\nu^2)+a_{1,\,2\lambda}\,\odot_{1,\,2\lambda}+a_{2,\,2\lambda}\,\odot_{2,\,2\lambda}+a_{4,\,2\lambda}\,\odot_{4,\,2\lambda}$$

$$+a_{5,\,2\lambda}\,\odot_{5,\,2\lambda}+\cdots$$

$$=(\lambda^2n^2-4\nu^2)+\tfrac{1}{2}\cos2\tau\odot_{1,\,2\lambda}-\frac{\nu}{\lambda n}\cos2\tau\odot_{2,\,2\lambda}-\frac{\nu\cos2\tau}{\lambda n}\odot_{4,\,2\lambda}$$

$$-\frac{2\nu^2}{\lambda^3n^3}\cos2\tau\odot_{5,\,2\lambda}-\cdots \qquad \cdots \quad [\mathrm{H}]$$

and

$$c=c_{1,\,2\lambda}\odot_{1,\,2\lambda}+c_{2,\,2\lambda}\odot_{2,\,2\lambda}+c_{4,\,2\lambda}\odot_{4,\,2\lambda}+c_{5,\,2\lambda}\odot_{5,\,2\lambda}+\cdots$$

$$=\frac{1}{4}\,\frac{\sin2\tau}{\lambda n}\,\odot_{1,\,2\lambda}-\frac{\nu\sin2\tau}{2\lambda^2n^2}\odot_{2,\,2\lambda}-\frac{\nu\sin2\tau}{2\lambda^3n^3}\odot_{4,\,2\lambda}-\frac{\nu\sin2\tau}{2\lambda^3n^3}\odot_{5,\,2\lambda}\quad[\mathrm{K}]$$

where, as already stated

$$a_{1,0}=\lambda^2n^2-4\nu^2 \text{ and } \cos\tau=0.$$

It is necessary to examine the expressions just obtained in order to see whether the complete integral of the equations [B] has been obtained.

The integer λ is determined so as *most nearly* to satisfy the relation

$$\odot_{1,0}=\lambda^2n^2-4\nu^2,$$

wherein everything excepting λ is known.

The negative value of λ will also satisfy the above relation.

Since $\cos\tau=0$ always, altogether there are *four* distinct values of τ obtainable *viz.*, $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$. Each of these will give a distinct value of c on substituting in [K] and different values for A and X. But fortunately, for our case c is always zero. Hence altogether we get *four* distinct solutions for A and X and these when multiplied by arbitrary constants will give the complete primitive of equations [B]. Such solutions as it is clear, will not contain the exponential factor.

Hence the solutions for ρ and σ are periodic functions and not pseudo-periodic as contemplated *a priori.*

(*b*) *The Particular Integral.*

We have now to determine the particular integral of equations [5] and [6] of Section III. We shall assume only one general term on the right-hand side and take the complete integral as the sum of a series of the corresponding solutions. The equations may therefore be written

$$\left.\begin{aligned}&\rho''-2\nu\sigma'+\rho\sum_{r=0}^{\infty}\odot_{1,r}\cos rnI+\sigma\sum_{r=1}^{\infty}\odot_{2,r}\sin rnI=\tfrac{1}{2}\odot_{3,m}e^{imnI}\\&\sigma''+2\nu\rho'+\rho\sum_{r=1}^{\infty}\odot_{4,r}\sin rnI+\sigma\sum_{r=1}^{\infty}\odot_{5,r}\cos rnI=0\end{aligned}\right\}\;\cdots\quad[\mathrm{B}']$$

Assume

$$\rho = e^{\iota mn\mathrm{I}}\,\mathrm{A},$$

$$\sigma = e^{\iota mn\mathrm{I}}\,\mathrm{X},$$

where A and X are as before functions of I.

Substituting these values of ρ and σ in [B′] we find

$$(\mathrm{A}'' + 2\iota mn\mathrm{A}' - m^2n^2\mathrm{A}) - 2\nu(\mathrm{X}' + \iota mn\mathrm{X}) + \mathrm{A}\sum_{r=0}^{\infty}\odot_{1,r}\cos rn\mathrm{I}$$

$$+\mathrm{X}\sum_{r=1}^{\infty}\odot_{2,r}\sin rn\mathrm{I} = \tfrac{1}{2}\odot_{3,m} \quad \ldots \quad [\mathrm{B}'\cdot 1]$$

$$(\mathrm{X}'' + 2\iota mn\mathrm{X}' - m^2n^2\mathrm{X}) + 2\nu(\mathrm{A}' + \iota mn\mathrm{A}) + \mathrm{A}\sum_{r=1}^{\infty}\odot_{1,r}\sin rn\mathrm{I}$$

$$+\mathrm{X}\sum_{r=1}^{\infty}\odot_{2,r}\cos rn\mathrm{I} = 0 \quad \ldots \quad [\mathrm{B}'\cdot 2]$$

Put

$$\mathrm{A} = \mathrm{A}_0 + \sum\mathrm{A}_{r,s}\odot_{r,s} + \sum\sum\mathrm{B}_{r,s,p,q}\,\odot_{r,s}\odot_{p,q} + \cdots,$$

$$\mathrm{X} = \mathrm{X}_0 + \sum\mathrm{X}_{r,s}\odot_{r,s} + \sum\sum\mathrm{Y}_{r,s,p,q}\,\odot_{r,s}\odot_{p,q} + \cdots,$$

in which $\odot_{1,0}$ is wanting. A_0, X_0 are constants, other coefficients of $\odot$'s are functions of I. Substitute these values of A and X in equations [B′·1] and [B′·2] and equate to zero the terms involving no $\odot$ except $\odot_{1,0}$. We have then

$$\left.\begin{aligned} -m^2n^2\mathrm{A}_0 - 2\nu\iota mn\mathrm{X}_0 + \mathrm{A}_0\odot_{1,0} &= \tfrac{1}{2}\odot_{3,m} \\ -m^2n^2\mathrm{X}_0 + 2\nu\iota mn\mathrm{A}_0 \qquad &= 0 \end{aligned}\right\} \quad \ldots \quad [\mathrm{B}'\cdot 3]$$

whence

$$\mathrm{A}_0 = \tfrac{1}{2}\odot_{3,m} \div (\odot_{1,0} - m^2n^2 + 4\nu^2)$$

$$\mathrm{X}_0 = \nu\iota\odot_{3,m} \div mn(\odot_{1,0} - m^2n^2 + 4\nu^2).$$

Equations [B′·1], [B′·2] can be fully written thus

$$(\sum\mathrm{A}''_{r,s}\odot_{r,s} + \ldots) + 2\iota mn(\sum\mathrm{A}'_{r,s}\odot_{r,s} + \cdots)$$

$$-m^2n^2(\mathrm{A}_0 + \sum\mathrm{A}_{r,s}\odot_{r,s} + \ldots)$$

$$-2\nu\{(\sum X'_{r,s}\odot_{r,s}+\ldots)+\iota mn(X_0+\sum X_{r,s}\odot_{,,s}+\quad)\}$$

$$+(A_0+\sum A_{r,s}\odot_{r,s}+\ldots)\sum_{0}^{\infty}\odot_{1,r}\cos rnI$$

$$+(X_0+\sum X_{r,s}\odot_{r,s}+\ldots)\sum_{1}^{\infty}\odot_{2,r}\sin rnI=\tfrac{1}{2}\odot_{3,m} \qquad \ldots \quad [C'\cdot 1]$$

$$(\sum X''_{r,s}\odot_{,,s}+\ldots)+2\iota mn(\sum X'_{r,s}\odot_{r,s}+\ldots)$$

$$-m^2n^2(X_0+\sum X_{r,s}\odot_{r,s}+\cdots)+2\nu\{(\sum A'_{r,s}\odot_{r,s}+\cdots)$$

$$+\iota mn(A_0+\sum A_{r,s}\odot_{r,s}+\cdots)\}+(A_0+\sum A_{,,s}\odot_{r,s}+\cdots).$$

$$\sum_{1}^{\infty}\odot_{4,r}\sin rnI+(X_0+\sum X_{r,s}\odot_{r,s}+\cdots)\sum_{1}^{\infty}\odot_{5,r}\cos rnI=0 \quad [D'\cdot 1]$$

Equate the coefficients of $\odot_{1,r}$ to zero :—

$$A''_{1,r}+2\iota mnA'_{1,r}-m^2n^2A_{1,r}-2\nu X'_{1,r}-2\nu\iota mnX_{1,r}$$

$$+A_{1,r}\odot_{1,0}+A_0\cos rnI=0 \qquad \ldots \quad [C'\cdot 2]$$

$$X''_{1,r}+2\iota mnX'_{1,r}-m^2n^2X_{1,r}+2\nu A'_{1,r}+2\nu\iota mnA_{1,r}=0 \qquad \ldots \quad [D'\cdot 2]$$

In the equations [C$'\cdot$2], [D$'\cdot$2] first put $e^{\iota rnI}$ then $e^{-\iota rnI}$ for $\cos rnI$. Finally, the complete solutions of them will be obtained by adding up and halving the results thus found out.

On solving we get

$$A_{1,r}=\tfrac{1}{2}A_0e^{\iota rnI}\div[(mn+rn)^2-\odot_{1,0}-4\nu^2]$$

$$+\tfrac{1}{2}A_0e^{-\iota rnI}\div[(mn-rn)^2-\odot_{1,0}-4\nu^2]$$

$$X_{1,r}=\nu\iota A_0e^{\iota rnI}\div[(mn+rn)\{(mn+rn)^2-\odot_{1,0}-4\nu^2\}]$$

$$+\nu\iota A_0e^{-\iota rnI}\div[(mn-rn)\{(mn-rn)^2-\odot_{1,0}-4\nu^2\}].$$

Now A_0 involves $\odot_{3,m}$, therefore $A_{1,r}\odot_{1,r}$, $X_{1,r}\odot_{1,r}$ each includes $\odot_{3,m}\odot_{1,r}$ as a factor. Hence these terms are negligible in comparison to A_0 and X_0.

Hence
$$\rho = e^{\iota mn\mathrm{I}}\,\mathrm{A}_0,$$
$$\sigma = e^{\iota mn\mathrm{I}}\,\mathrm{X}_0,$$

where A_0, X_0 are given above. Now put $\rho = e^{-\iota mn\mathrm{I}}\,\mathrm{A}_0$, $\sigma = e^{-\iota mn\mathrm{I}}\,\mathrm{X}_0$ and we get the same values for A_0 and X_0.

Next put
$$\frac{\mathrm{I}}{2\iota}\,\odot_{6,m}e^{\iota mn\mathrm{I}},$$

in the right-hand member of the second equation in [B′], and zero in the right-hand member of the first equation in [B′]. We get the equations

$$\left.\begin{aligned}&\rho'' - 2\nu\sigma' + \rho\sum_{r=0}^{\infty}\odot_{1,r}\cos rn\mathrm{I} + \sigma\sum_{r=1}^{\infty}\odot_{2,r}\sin rn\mathrm{I} = 0\\ &\sigma'' + 2\nu\rho' + \rho\sum_{r=0}^{\infty}\odot_{4,r}\sin rn\mathrm{I} + \sigma\sum_{r=1}^{\infty}\odot_{5,r}\cos rn\mathrm{I} = \frac{1}{2\iota}\odot_{6,m}e^{\iota mn\mathrm{I}}\end{aligned}\right\}\quad[\mathrm{B}'']$$

Proceeding in the way mapped out as *ante*, we find

$$\mathrm{A}_0 = -\nu mn\odot_{6,m} \div [m^2n^2(\odot_{1,0} - m^2n^2 + 4\nu^2)],$$

$$\mathrm{X}_0 = \frac{1}{2\iota}\,\odot_{6,m}(\odot_{1,0} - m^2n^2) \div m^2n^2[\odot_{1,0} - m^2n^2 + 4\nu^2].$$

As before all other terms in A, **X** are negligible, so we need not calculate them. For the complete solution we should calculate the corresponding terms for

$$-\frac{1}{2\iota}e^{-\iota mn\mathrm{I}},$$

which are easily obtained from those involving

$$\frac{1}{2\iota}e^{\iota mn\mathrm{I}}.$$

Thus

$$\mathrm{A}_0 = -\nu mn\odot_{6,m} \div [m^2n^2(\odot_{1,0} - m^2n^2 + 4\nu^2)],$$

$$\mathrm{X}_0 = -\frac{1}{2\iota}\,\odot_{6,m}(\odot_{1,0} - m^2n^2) \div [m^2n^2(\odot_{1,0} - m^2n^2 + 4\nu^2)],$$

Hence

$$\rho=\sum_{m=0}^{\infty}[mn\odot_{3,m}-2\nu\odot_{6,m}]\cos mn\mathrm{I}\div mn[\odot_{1,0}-m^2n^2+4\nu^2],$$

$$\sigma=\sum_{m=0}^{\infty}[-2\nu\odot_{3,m}+mn\odot_{6,m}(\odot_{1,0}-m^2n^2)]\sin mn\mathrm{I}$$

$$\div mn[\odot_{1,0}-m^2n^2+4\nu^2].$$

V

SUMMARY AND CONCLUSION.

The results of the above analysis may be thus briefly summarised :— The perturbed orbit may be represented by

$$r=b(1+\rho)$$

$$\theta=(\omega'-\omega)t+a+\sigma$$

where b=radius of the unperturbed circular orbit,

ω'=angular velocity of rotation of the outer electron,

ω=............inner......,

a=any arbitary epoch,

ρ and σ are elements of perturbation. Tho above analysis shows that they are both periodic functions of $(\omega'-\omega)t$. The method adopted is that due to Goldsbrough who introduced a modification of the procedure in the theory of lunar perturbations first initiated by Hill and developed his results on the lines mapped out by Whittaker, Young and others.

In atomic problems, the interest does not lie in calculating the exact position of the satellite at different times as in the case of lunar motion. The problem is to quantize the orbits and to find out if from such quantized orbits, the energy can be calculated as a function of n and k, the total quantum number and the azimuthal quantum number respectively; and then to verify this energy with the spectral terms *mp*, *md* etc.

Hitherto, the quantization has been confined to very simple orbits—such as circular orbits by Bohr and elliptic orbits by Sommerfeld. Epstein[1] discussed the case of orbits subjected to the perturbations due to a uniform field and gave an explanation of the Stark effect; but Nicholson[2] finds that the method is not mathematically sound.

Hicks[3] has recently raised an important objection to Sommerfeld's principle that in all *mp* and *md* orbits, $\int p_\theta \delta q_\theta = 2h$ and $3h$ respectively.

The above discussion shows that the handling of the general problem is much difficult than can be imagined. I have not yet succeeded in quantizing perturbed orbits, and therefore cannot say how far these investigations will support Sommerfeld's general theorem. This is in the course of my investigation.

A glance at the values of the several constants $A_{r,s}$, $X_{r,s}, \ldots$, shows that the perturbed motion constitutes an ensemble of discrete harmonic oscillations having different frequencies. So far as the radial perturbed element ρ is concerned, it is easy to see, we must have a *range of vibrations* within the maximum and minimum. Under such circumstances, at any rate, we must expect that the perturbed system will not possess any sharply separated *stationary states*. The compound motion has rigorously a two-fold periodic character,— *one*, round the kernel in a closed periodic orbit for the unperturbed system *i.e.*, neglecting the *zusatzfeld*, *two*, *librations*—both radial and azimuthal — of the electron about the position it would have occupied at any instant for the unperturbed system, due to the quota of perturbing forces subjected to it by the *zusatzfeld* calculated in Sec. II.

So corresponding to a single stationary state in the unperturbed system there exists a multiple of slowly varied stationary states in the perturbed system, possessing a pronounced cycle ; of course, the resultant frequency of the group of perturbation oscillations must be vanishingly small as compared with the time of revolution of the electron in the undisturbed state. But whether or not the motion is what is technically called conditionally periodic is difficult to judge *a priori*.

Bohr has laid down[4] that for a transition between two of the states corresponding to the perturbed system a radiation is emitted " whose frequency stands in the same relationship to the periodic course of the variations in the orbit, as the spectrum of a simple periodic system does to its motion in the stationary states." Any more, quantization is possible by exhibiting a new phase of the adiabatic hypothesis first propounded by Ehrenfest[1], or what is strictly called the principle of "mechanical transformability" of stationary states. In that case, however, there is an *a priori* probability of getting an almost identical series formula as obtained by Sommerfeld. Nevertheless, it is undesirable at this stage to try to incorporate an analysis and posit a principle having a feature somewhat foreign to what has been set forth hereto. This is deferred to a future occasion.

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[1] Sommerfeld, '*Atombau und Spektrallinien*', Third ed., pp. 339-51.

[2] '*Phil. Mag.*', July, 1922.

[3] '*Phil. Mag.*', Aug., 1922.

[4] Theory of spectra and Atomic constitution, p. 39.

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[1] Proc. Acad. Amsterdam, XVI, p. 591 (1914), *Phy. Zeitschr.* XV. p. 657 (1914), *Ann. d. Phys.* LI, p. 827 (1916), Phil. Mag. XXXIII, p. 500 (1917).

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ALGEBRA OF POLYNOMINALS

BY

NRIPENDRANATH GHOSH

Chapter II

Expansions

12. The problem of expanding a given explicit function of a polynomial or a number of polynomials (and their derivatives) admits of an elegant treatment by means of the theorems established in the preceding chapter. The expansions obtained are of highly general character and cases may occur where these expansions fail to be consistent when numerical values are substituted for the variables involved. We shall not attempt to enquire into the validity of such expansions, but on the other hand, assume those conditions to be existing under which the expansions are arithmetically intelligible.

13. Let then $\phi(u_\alpha)$, in Art 4, be expanded in a series of ascending powers of z of the form

$$A_0+A_1z+A_2z^2+\cdots+A_rz^r+\cdots,$$

then since

$$\frac{d}{dz}\phi(u_\alpha)=\Delta_{\alpha 0}\phi(u_\alpha),$$

we must have

$$\frac{d}{dz}(A_0+A_1z+A_2z^2+\cdots)=\Delta_{\alpha 0}(A_0+A_1z+A_2z^2+\cdots),$$

or $$A_1+2A_2z+3A_3z^2+\cdots=\Delta_{\alpha 0}A_0+\Delta_{\alpha 0}A_1z+\Delta_{\alpha 0}A_2z^2+\cdots;$$

whence comparing coefficients,

$$A_1 = \Delta_{x0} A_0,$$

$$2A_2 = \Delta_{x0} A_1,$$

$$3A_3 = \Delta_{x0} A_2,$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$(r+1)A_{r+1} = \Delta_{x0} A_r,$$

or when reduced

$$A_1 = \Delta_{x0} A_0,$$

$$A_2 = \frac{\Delta^2_{x0}}{\underline{|2}} A_0,$$

$$A_3 = \frac{\Delta^3_{x0}}{\underline{|3}} A_0,$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$A_{r+1} = \frac{\Delta^{r+1}_{x0}}{\underline{|r+1}} A_0;$$

where A_0 is evidently equal to $\phi(a_0)$.

Thus by successive application of the operator Δ_{x0} we have a means of calculating all the coefficients in the expansion of $\phi(u_x)$.

14. It can be inferred from the following typical calculations that any coefficient A_r (in above) is a linear homogeneous function of $\phi'(a_0), \phi''(a_0), \cdots \phi^{(r)}(a_0)$ only. the coefficient of any derivative $\phi^{(t)}(a_0)$ in A_r being a rational and integral function of degree t and weight r ($t \not> r$) involving $a_1, a_2, a_3, \cdots a_r$ only of the coefficients of u_x.

We have

$$A_1 = a_1\phi'(a_0),$$

$$\underline{|2}\, A_2 = a_1^2\phi''(a_0) + 2a_2\phi'(a_0),$$

$$|3\, A_3 = a_1^3\phi'''(a_0) + 6a_1a_2\phi''(a_0) + 6a_3\phi'(a_0),$$

$$\underline{|4}\, A_4 = a_1^4\phi^{iv}(a_0) + 12a_1^2a_2\phi'''(a_0) + (12a_2^2 + 24a_1a_3)\phi''(a_0) + 24a_4\phi'(a_0),$$

$$\underline{|5}\, A_5 = a_1^5\phi^{v}(a\ \ ) + 20a_1^3a_2\phi^{iv}(a_0) + (60a_1^2a_3 + 60a_1a_2^2)\phi'''(a_0) + (120a_1a_4 + 120a_2a_3)\phi''(a_0) + 120a_5\phi'(a_0).$$

The coefficients A's in the expansion of $\phi(u_a)$ are connected by means of the operator Δ_{a_0}. This is, however, not the only connection existing among those coefficients. There are others and we proceed to find them.

15. Differential relations among the coefficients in the expansion of $\phi(u_a)$:—

We have

$$\frac{\partial}{\partial a_r}\phi(u_a) = \frac{\partial\phi}{\partial u_a}\cdot\frac{\partial u_a}{\partial a_r} = \frac{\partial\phi}{\partial u_a}\cdot z^r = z^r\frac{\partial}{\partial a_0}\phi(u_a),$$

and this holds true for all values $_{0,\ 1,\ 2,\ 3,}\cdots{}_n$ of r.

Since $\phi(u_a)$ is expanded in the form

$$A_0 + A_1z + A_2z^2 + \cdots + A_rz^r$$

we must have by the above identity

$$\frac{\partial}{\partial a_r}(A_0 + A_1z + A_2z^2 + \cdots) = z^r\frac{\partial}{\partial a_0}(A_0 + A_1z + A_2z^2 + \cdots);$$

whence comparing coefficients

$$\frac{\partial A_0}{\partial a_r} = \frac{\partial A_1}{\partial a_r} = \frac{\partial A_2}{\partial a_r} = \cdots = \frac{\partial A_{r-1}}{\partial a_r} = 0,$$

$$\frac{\partial A_r}{\partial a_r} = \frac{\partial A_0}{\partial a_0},$$

$$\frac{\partial A_{r+1}}{\partial a_r} = \frac{\partial A_1}{\partial a_0},$$

$$\frac{\partial A_{r+2}}{\partial a_r} = \frac{\partial A_2}{\partial a_0},$$

and so on ; where r may have any of the values $_{0},\ _{1},\ _{2},\ _{3},\cdots n$.

16. These differential relations simplify the process of operation by Δ_{a_0} upon the coefficients A's. Let us take from Art 13, the equation

$$(r+1)A_{r+1} = \Delta_{a_0} A_r,$$

i.e.,

$$(r+1)A_{r+1} = \left(a_1 \frac{\partial}{\partial a_0} + 2a_2 \frac{\partial}{\partial a_1} + 3a_3 \frac{\partial}{\partial a_2} + + \cdots + na_n \frac{\partial}{\partial a_{n-1}} \right) A_r$$

$$= a_1 \frac{\partial A_r}{\partial a_0} + 2a_2 \frac{\partial A_r}{\partial a_1} + 3a_3 \frac{\partial A_r}{\partial a_2}$$

$$+ \cdots (r+1) a_{r+1} \frac{\partial A_r}{\partial a_r}, \qquad \text{(if } r<n\text{)}$$

$$= a_1 \frac{\partial A_r}{\partial a_0} + 2a_2 \frac{\partial A_{r-1}}{\partial a_0} + 3a_3 \frac{\partial A_{r-2}}{\partial a_0}$$

$$+ \cdots (r+1) a_{r+1} \frac{\partial A_0}{\partial a_0}, \qquad \text{(by Art 15)}$$

$$= \frac{\partial}{\partial a_0} (a_1 A_r + 2a_2 A_{r-1} + 3a_3 A_{r-2} + \cdots (r+1) a_{r+1} A_0).$$

The above also holds good if $r=$ or $>n$.

17. By means of the identity in Art 6, we get further relations among the coefficients A's.

Since

$$z^2\frac{d}{dz}\phi(u_a)=\left(\Delta_{a_2}+na_n z\frac{\partial}{\partial a_n}\right)\phi(u_a),$$

we must have

$$z^2\frac{d}{dz}(A_0+A_1z+A_2z^2+\cdots)=\Delta_{a_2}(A_0+A_1z+A_2z^2+\cdots)$$

$$+na_n z\frac{\partial}{\partial a_n}(A_0+A_1z+A_2z^2+\cdots),$$

or

$$z^2(A_1+2A_2z+3A_3z^2+\cdots)=\Delta_{a_2}(A_0+A_1z+A_2z^2+\cdots)$$

$$+na_n z^{n+1}\frac{\partial}{\partial a_0}(A_0+A_1z+A_2z^2+\cdots):\quad \text{(by Art 15)}$$

Whence comparing coefficients

$$\left.\begin{array}{l}\Delta_{a_2}A_0=0,\\ \Delta_{a_2}A_1=0,\\ \Delta_{a_2}A_2=A_1,\\ \Delta_{a_2}A_3=2A_2,\\ \cdots\qquad\cdots\\ \cdots\qquad\cdots\\ \Delta_{a_2}A_n=(n-1)A_{n-1},\end{array}\right\}\quad\begin{array}{l}\Delta_{a_2}A_{n+1}+na_n\dfrac{\partial A_0}{\partial a_0}=nA_n,\\[2ex] \Delta_{a_2}A_{n+2}+na_n\dfrac{\partial A_1}{\partial a_0}=(n+1)A_{n+1},\\[2ex] \Delta_{a_2}A_{n+3}+na_n\dfrac{\partial A_2}{\partial a_0}=(n+2)A_{n+2},\end{array}$$

and so on.

These relations may be regarded as reciprocal to those in Art 13.

18. Allied expansions :—

There are other allied forms in which $\phi(u_a)$ may be expanded. The calculation of the coefficients in these expansions may be made to depend on the fundamental one in Art 13. The forms of these allied expansions are given below :—

(1) $\phi(a_0+a_1z)+A'_2z^2+A'_3z^3+A'_4z^4+\cdots$

(2) $\phi(a_0+a_1z+a_2z^2)+A''_3z^3+A''_4z^4+\cdots$

(3) $\phi(a_0+a_1z+a_2z^2+a_3z^3)+A'''_4z^4+\cdots$

and so on.

Let us find A'_r in the first of these allied forms. We observe that A'_r must be a part of A_r. To specify that part we notice that A'_r vanishes when

$$a_2 = a_3 = a_4 = \cdots = a_r = 0\,;$$

so that A'_r is the residue of A_r left by removing that part which is not equal to zero when

$$a_2 = a_3 = a_4 = \cdots = a_r = 0.$$

Similar remark applies to other allied forms.

19. Expansion of a function involving a number of polynomials :—

Let $\phi(u_a, u_b, u_c \cdots)$, in Art 7, be expanded in the form

$$(A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots,$$

then since

$$\frac{d}{dz}\phi(u_a, u_b, u_c \cdots) = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} + \cdots)\phi(u_a, u_b, u_c \cdots),$$

we must have

$$\frac{d}{dz}\{(A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots\}$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots)\{(A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots\},$$

or $(A)_1 + 2(A)_2 z + 3(A)_3 z^2 + \cdots$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots)\{(A)_0 + (A)_1 z + (A)_2 z^2 + \cdots\}.$$

Representing the compound operator $\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots$ by $(\Delta)_0$ and comparing the coefficients of like powers of z we have

$$(A)_1 = (\Delta)_0 (A)_0,$$

$$2(A)_2 = (\Delta)_0 (A)_1,$$

$$3(A)_3 = (\Delta)_0 (A)_2,$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$(r+1)(A)_{r+1} = (\Delta)_0 (A)_r\,;$$

or when reduced

$$(A)_1=(\Delta)_0(A)_0,$$

$$(A)_2=\frac{(\Delta)_0^2}{\underline{|2}}(A)_0,$$

$$(A)_3=\frac{(\Delta)_0^3}{\underline{|3}}(A)_0,$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$(A)_{r+1}=\frac{(\Delta)_0^{r+1}}{\underline{|r+1}}(A)_0,$$

where $(A)_0$ is evidently equal to $\phi(a_0,b_0,c_0,\cdots)$.

Thus by successive application of the operator $(\Delta)_0$ we have a means of calculating all the coefficients in the expansion of $\phi(u_a,u_b,u_c\ldots)$.

20. Differential relations among the coefficients in the expansion of $\phi(u_a,u_b,u_c\cdots)$:—

We have

$$\frac{\partial}{\partial a_r}\phi(u_a,u_b,u_c\cdots)=\frac{\partial\phi(u_a,u_b,u_c\cdots)}{\partial u_a}\cdot\frac{\partial u_a}{\partial a_r}$$

$$=z^r\,\frac{\partial\phi(u_a,u_b,u_c\cdots)}{\partial u_a}=z^r\,\frac{\partial}{\partial a_0}\phi(u_a,u_b,u_c\cdots)\,;$$

and this holds true for all values $_0,\ _1,\ _2,\ _3\cdots n$ of r. Similarly

$$\frac{\partial}{\partial b_q}\phi(u_a,u_b,u_c\cdots)=z^q\,\frac{\partial}{\partial b_0}\phi(u_a,u_b,u_c\cdots)\,;$$

which holds true for all values $_0,\ _1,\ _2,\ _3\cdots m$ of q;

$$\frac{\partial}{\partial c_p}\phi(u_a,u_b,u_c\cdots)=z^p\,\frac{\partial}{\partial c_0}\phi(u_a,u_b,u_c\cdots)\,;$$

which holds true for all values $_0,\ _1,\ _2,\ _3\cdots l$ of p; and so on.

Referring to Art 15, the differential relations among the coefficients (A), may be obtained with regard to each of the variables a's, b's, c's $\cdots$ from the identities above.

By means of these differential relations $(r+1)(A)_{r+1}$ may be expressed in the form

$$\frac{\partial}{\partial a_0}\{a_1(A)_r+2a_2(A)_{r-1}+3a_3(A)_{r-2}+\cdots+(r+1)a_{r+1}(A)_0\}$$

$$+\frac{\partial}{\partial b_0}\{b_1(A)_r+2b_2(A)_{r-1}+3b_3(A)_{r-2}+\cdots+(r+1)b_{r+1}(A)_0\}$$

$$+\frac{\partial}{\partial c_0}\{c_1(A)_r+2c_2(A)_{r-1}+3c_3(A)_{r-2}+\cdots+(r+1)c_{r+1}(A)_0\}$$

$$+\cdots$$

21. When the relative magnitudes of $l, m, n\cdots$ are given it is possible to obtain, by means of art 8, further relations among the coefficients (A)'s by proceeding exactly in the same way as in Art 17.

There is a set of allied forms in which $\phi(u_a,u_b,u_c\ldots)$ may be expanded. The coefficients in each of these allied expansions may be deduced from those in the fundamental one.

22. Expansion of a function involving a polynomial and its derivatives :—

Let

$$\phi(u_a,u'_a,u''_a,\ldots u_a^{(r)}),$$

in Art 9, be expanded in the form

$$\bar{A}_0+\bar{A}_1z+\bar{A}_2z^2+\ldots,$$

then since

$$\frac{d}{dz}\phi(u_a,u'_a,u''_a\ldots u_a^{(r)})=\Delta_{a0}\phi(u_a,u'_a,u''_a\ldots u_a^{(r)}),$$

we must have

$$\frac{d}{dz}(\bar{A}_0+\bar{A}_1z+\bar{A}_2z^2+\bar{A}_3z^3+\ldots)$$

$$=\Delta_{a0}(\bar{A}_0+\bar{A}_1z+\bar{A}_2z^2+\ldots),$$

or
$$\bar{A}_1+2\bar{A}_2z+3\bar{A}_3z^2+\dots$$
$$=\Delta_{\alpha 0}\bar{A}_0+\Delta_{\alpha 0}\bar{A}_1z+\Delta_{\alpha 0}\bar{A}_2z^2+\dots;$$

whence comparing coefficients,

$$\bar{A}_1=\Delta_{\alpha 0}\bar{A}_0,$$
$$2\bar{A}_2=\Delta_{\alpha 0}\bar{A}_1,$$
$$3\bar{A}_3=\Delta_{\alpha 0}\bar{A}_2,$$

and so on, where $\bar{A}_0$ is evidently equal to

$$\phi(a_0,a_1,\underline{|2}a_2,\underline{|3}a_3,\cdots\underline{|r}a_r).$$

The coefficients $\bar{A}$'s are connected only by $\Delta_{\alpha 0}$. It has not yet been possible to find other connections existing among them.

23. Expansion of a transformed polynomial :—

Let $u_\alpha(\psi t)$, the transformed polynomial of $u_\alpha(z)$, in art 11, be expanded in a series of ascending powers of t of the form

$$a_0+a_1t+a_2t^2+\cdots,$$

then since

$$\frac{1}{\psi'(t)}\ \frac{d}{dt}\{u_\alpha(\psi t)\}=\Delta_{\alpha 0}\{u_\alpha(\psi t)\},$$

we must have

$$\frac{1}{\psi'(t)}\ \frac{d}{dt}(a_0+a_1t+a_2t^2+\cdots)$$
$$=\Delta_{\alpha 0}(a_0+a_1t+a_2t^2+\cdots),$$

or
$$a_1+2a_2t^2+3a_3t^2+\cdots$$
$$=\psi'(t)\Delta_{\alpha 0}(a_0+a_1t+a_2t^2+\cdots)$$
$$=\Delta_{\alpha 0}\psi'(t)\{a_0+a_1t+a_2t^2+\cdots).$$

$\psi'(t)$ being known from the given transformation $z=\psi(t)$ (it is usual to restrict $\psi'(t)$ to rational integral functions alone) we can express the right-hand side of the above identity in a series of ascending powers of t. Now comparing coefficients of like powers of t the coefficients a's may be obtained. a_0 is evidently equal to $u_\alpha(\psi 0)$.

If $\psi(t)$ be a rational and integral function of kth degree in t, the transformation is one of the kth order. If, moreover, $\psi(0)=0$, the transformation is called a simple transformation of the kth order.

A function of the transformed polynomial may similarly be expanded.

24. Polynomials of degree infinite :—

When the degree n of the polynomial u_a increases without limit it becomes tho polynomial v_a of degree infinite. For finite numerical values of the variables such a polynomial may have an infinite value and the polynomial is said to be divergent (for those values of the variables). Otherwise the polynomial is said to be convergent.

We may extend (with necessary changes) the theorems of the last chapter and those of the present one to include polynomials of degree infinite provided initially they are convergent.

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ON THE MOTION OF A VISCOUS LIQUID BETWEEN TWO NON-CONCENTRIC CIRCULAR CYLINDERS.

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INTRODUCTION.

1. As far as I know, the problem of translation of two non-concentric infinite circular cylinders in a viscous liquid has not been investigated by any writer, though a solution of the analogous problem of rotation of two circular cylinders in a viscous liquid has been given in a recent issue [1] of the Proceedings of the Royal Society, by Dr. G. B. Jeffery.

In the present paper, I have discussed the problem of translation of two parallel infinite circular cylinders in a viscous liquid. The solution is different in form according as one cylinder does or does not enclose the other. In the former case the problem can be solved in finite terms and we shall get the current function of the "initial motion"; while in the latter case the problem is in general insoluble, that is to say, except in special circumstances, "there is no steady motion which satisfies all the necessary conditions."

THE CURRENT-FUNCTION.

2 Let

$$x+iy=c\tan\tfrac{1}{2}(\xi+i\eta)$$

Then

$$x=c\,\frac{\sin\xi}{\cosh\eta+\cos\xi},\quad y=\frac{c\sinh\eta}{\cosh\eta+\cos\xi}.$$

$$h^2=\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2$$

$$=\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{dy}\right)^2$$

$$=c^{-2}(\cosh\eta+\cos\xi)^2.$$

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[1] *The Rotation of two Circular Cilenders in a Viscous Flind,* Proc. Roy. Soc., Vol. A. 101, No. A. 709, (1922). p. 169.

and

$$r^2 = c^2(\cosh\eta - \cos\xi)/(\cosh\eta + \cos\xi)$$

The current-function satisfies the equation

$$\nabla^4\psi = 0$$

To find a solution, let us write

$$\psi = H_1 \sin\xi + H_2 \sin\xi/(\cosh\eta + \cos\xi)$$

$$c^2\nabla^2\psi = \sin\xi\{(H_1'' - H_1)(\cosh\eta + \cos\xi)^2 + H_2''(\cosh\eta + \cos\xi)$$

$$-2H_2' \sinh\eta\}$$

Operating on $c^2\nabla^2\psi$ by δ^2 which stands for $\dfrac{\partial^2}{\partial\xi^2} + \dfrac{\partial^2}{\partial\eta^2}$,

$$c^2\delta^2\nabla^2\psi = \sin\xi\{(H_1^{iv} - H_1'')(\cosh\eta + \cos\xi)^2 + 4(H_1''' - H_1')\sinh\eta \times$$
$$(\cosh\eta + \cos\xi) + (H_1'' - H_1)\Big(2\cosh 2\eta + 2\cosh\eta\cos\xi - (\cosh\eta + \cos\xi)^2$$
$$-6(\cosh\eta + \cos\xi)\cos\xi + 2(1 - \cos^2\xi)\Big) + H_2^{iv}(\cosh\eta + \cos\xi) - 4H_2''\cos\xi$$
$$-4H_2''\cosh\eta\}$$

Equating to zero the coefficients of the several powers of cos ξ, we obtain the following equations

$$H_1^{iv} - 10H_1'' + 9H_1 = 0 \qquad \ldots \qquad (1)$$

$$2(H_1^{iv} - H_1'')\cosh\eta + 4(H_1''' - H_1')\sinh\eta - 6(H_1'' - H_1)\cosh\eta$$
$$+ H_2^{iv} - 4H_2'' = 0 \qquad \ldots \qquad (2)$$

$$(H_1^{iv} - H_1'')\cosh^2\eta + 4(H_1''' - H_1')\sinh\eta\cosh\eta$$
$$+3(H_1'' - H_1)\cosh^2\eta + H_2^{iv}\cosh\eta - 4H_2''\cosh\eta = 0 \qquad \ldots \qquad (3)$$

The third equation is not independent but follows directly from the first two.

Solving we get

$$H_1 = (A\cosh 3\eta + B\sinh 3\eta + C\cosh\eta + D\sinh\eta)$$

$$H_2 = (-\tfrac{1}{2}A\cosh 4\eta - \tfrac{1}{2}B\sinh 4\eta + E\cosh 2\eta + F\sinh 2\eta + G\eta + H)$$

Therefore the stream-function can be written in the form

$$\psi=(\mathrm{A}\cosh 3\eta+\mathrm{B}\sinh 3\eta+\mathrm{C}\cosh\eta+\mathrm{D}\sinh\eta)\sin\xi$$

$$+\left(-\frac{1}{2}\mathrm{A}\cosh 4\eta-\frac{1}{2}\mathrm{B}\sinh 4\eta+\mathrm{E}\cosh 2\eta+\mathrm{F}\sinh 2\eta\pm\mathrm{G}\eta+\mathrm{H}\right)$$

$$\times\frac{\sin\xi}{(\cosh\eta+\cos\xi)}$$

Let the two cylinders be defined by constant values of η say $\eta=\alpha$, $\eta=\beta$. We may take α positive and greater than β, then β will be positive or negative according as the first cylinder does or does not enclose the second.

Let the outer cylinder be moved with velocity V_1 and the inner one with velocity V_2 parallel to the axis of Y.

If we write

$$u=-\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\psi}{\partial x}$$

then since at the surface of the cylinder

$$u=0,\ v=\mathrm{V}_1,$$

the boundary conditions become, when $\eta=\alpha$

$$\frac{\partial\psi}{\partial\xi}=\mathrm{V}_1 c\left\{\frac{\cos\xi}{(\cosh\alpha+\cos\xi)}+\frac{\sin^2\xi}{(\cosh\alpha+\cos\xi)^2}\right\}$$

$$\frac{\partial\psi}{\partial\eta}=-\mathrm{V}_1 c\left\{\frac{\sin\xi\sinh\alpha}{(\cosh\alpha+\cos\xi)^2}\right\}\qquad\ldots\qquad(a)$$

and when $\eta=\beta$

$$\frac{\partial\psi}{\partial\xi}=\mathrm{V}_2 c\left\{\frac{\cos\xi}{(\cosh\beta+\cos\xi)}+\frac{\sin^2\xi}{(\cosh\beta+\cos\xi)^2}\right\}$$

$$\frac{\partial\psi}{\partial\eta}=-\mathrm{V}_2 c\left\{\frac{\sin\xi\sinh\beta}{(\cosh\beta+\cos\xi)^2}\right\}\qquad\ldots\qquad(a')$$

But when $\eta=\alpha$, we get from the expression for the stream function

$$\frac{\partial\psi}{\partial\xi}=\{A\cosh 3\alpha+B\sinh 3\alpha+C\cosh\alpha+D\sinh\alpha\}\cos\xi$$

$$+\left\{-\frac{1}{2}A\cosh 4\alpha-\frac{1}{2}B\sinh 4\alpha+E\cosh 2\alpha+F\sinh 2\alpha+G\alpha+H\right\}$$

$$\times\left\{\frac{\sin^2\xi}{(\cosh\alpha+\cos\xi)^2}+\frac{\cos\xi}{(\cosh\alpha+\cos\xi)}\right\}$$

$$\frac{\partial\psi}{\partial\eta}=\{3\,A\sinh 3\alpha+3\,B\cosh 3\alpha+C\sinh\alpha+D\cosh\alpha\}\sin\xi$$

$$+\{-2\,A\sinh 4\alpha-2B\cosh 4\alpha+2E\sinh 2\alpha+2F\cosh 2\alpha+G\}$$

$$\times\frac{\sin\xi}{(\cosh\alpha+\cos\xi)}-\left\{-\frac{1}{2}A\cosh 4\alpha-\frac{1}{2}B\sinh 4\alpha+E\cosh 2\alpha\right.$$

$$\left.+F\sinh 2\alpha+G\alpha+H\right\}\frac{\sin\xi\sinh\alpha}{(\cosh\alpha+\cos\xi)^2}. \qquad \ldots \qquad (b)$$

From (a) and (b) we get the following equations

$$A\cosh 3\alpha+B\sinh 3\alpha+C\cosh\alpha+D\sinh\alpha=0 \qquad \ldots \qquad (4)$$

$$-\frac{1}{2}A\cosh 4\alpha-\frac{1}{2}B\sinh 4\alpha+E\cosh 2\alpha+F\sinh 2\alpha$$

$$+G\alpha+H=V_1c \qquad \ldots \qquad (5)$$

$$3A\sinh 3\alpha+3B\cosh 3\alpha+C\sinh\alpha+D\cosh\alpha=0 \qquad \ldots \qquad (6)$$

$$-2A\sinh 4\alpha-2B\cosh 4\alpha+2E\sinh 2\alpha+2F\cosh 2\alpha+G=0 \qquad (7)$$

together with four precisely similar equations obtained from these by writing β and V_2 for α and V_1.

Solving, we have,

$$A=B=C=D=0.$$

$$E=\frac{(V_1-V_2)c}{2}\quad\frac{(\cosh 2\alpha-\cosh\ 2\beta)}{(\alpha-\beta)\sinh 2(\alpha-\beta)+1-\cosh 2(\alpha-\beta)}$$

$$F=\frac{(V_1-V_2)c}{2}\quad\frac{(\sinh 2\beta-\sinh 2\alpha)}{(\alpha-\beta)\sinh 2(\alpha-\beta)+1-\cosh 2(\alpha-\beta)}$$

$$G=(V_1-V_2)c\quad\frac{\sinh 2(\alpha-\beta)}{(\alpha-\beta)\sinh 2(\alpha-\beta)+1-\cosh 2(\alpha-\beta)}$$

$$H=-\left\{\frac{-2(V_1+V_2)c+2(V_1+V_2)c\cosh 2\ (\alpha-\beta)+4c'\beta V_1}{4(\alpha-\beta)\sinh 2(\alpha-\beta)+4-4\cosh 2(\alpha-\beta)}\right.$$

$$\underline{-\alpha V_2)\sinh 2(\alpha-\beta)}$$

Substituting tho values of the constants we can obtain the stream-function.

THE PRESSURE.

3. We know that $\mu\nabla^2\psi$ and p are conjugate functions.

Now

$$c^2\nabla^2\psi=2E\ (\sin 2\xi\cosh 2\eta+2\sin\xi\cosh\eta)$$

$$+2F\ (\sin 2\xi\sinh 2\eta+2\sin\xi\sinh\eta)-2G\sin\xi\sinh\eta$$

Therefore

$$p=\frac{\mu}{c^2}\ \{2E\ (\cos 2\xi\sinh 2\eta+2\cos\xi\sinh\eta)$$

$$+2F\ (\cos 2\xi\cosh 2\eta+2\ \cos\xi\cosh\eta)-2G\cos\xi\cosh\eta\}+\text{constant}.$$

THE RESISTANCE.

4. The formulæ for the elongation of the shear are

$$e=-f=\frac{1}{2}\left(\frac{\partial h^2}{\partial\xi}\ \frac{\partial\psi}{\partial\eta}+\frac{\partial h^2}{\partial\eta}\ \frac{\partial\psi}{\partial\xi}\right)+h^2\frac{\partial^2\psi}{\partial\xi\partial\eta},$$

$$\gamma=h^2\left(\frac{\partial^2\psi}{\partial\eta^2}-\frac{\partial^2\psi}{\partial\xi^2}\right)+\frac{\partial h^2}{\partial\eta}\ \frac{\partial\psi}{\partial\eta}-\frac{\partial h^2}{\partial\xi}\ \frac{\partial\psi}{\partial\xi}$$

(Ibbetson, *Elasticity.*)

Substituting, we get

$$e=f=o.$$

When $\eta=\alpha$,

$$\gamma=c^{-2}(\cosh\alpha+\cos\xi)(4\mathrm{E}\cosh 2\alpha+4\mathrm{F}\sinh 2\alpha)\sin\xi$$

and when $\eta=\beta$,

$$\gamma=c^{-2}(\cosh\beta+\cos\xi)(4\mathrm{E}\cosh 2\beta+4\mathrm{F}\sinh 2\beta)\sin\xi$$

The resistance acting on the outer cylinder is given by

$$\mathrm{R}_1=\int-\left(\mathrm{U}\frac{dy}{ds}+p\frac{dx}{ds}\right)ds$$

where $\mathrm{U}=\mu\gamma$, and the integration is taken round the circle.

$$\mathrm{R}_1=\frac{\pi\mu}{c}\{2(\mathrm{E}\sinh 4\alpha+\mathrm{F}\cosh 4\alpha)-4(\mathrm{E}\sinh 2\alpha$$

$$+\mathrm{F}\cosh 2\alpha)+2\mathrm{F}+2\mathrm{G}(\cosh 2\alpha+1)\}$$

$$=4\pi\mu(\mathrm{V}_1-\mathrm{V}_2)\frac{\sinh 2(\alpha-\beta)}{\{(\alpha-\beta)\sinh 2(\alpha-\beta)+1-\cosh 2(\alpha-\beta)\}}.$$

Similarly the resistance acting on the inner cylinder is given by

$$\mathrm{R}_2=-4\pi\mu(\mathrm{V}_1-\mathrm{V}_2)\frac{\sinh 2(\alpha-\beta)}{\{(\alpha-\beta)\sinh 2(\alpha-\beta)+1-\cosh 2(\alpha-\beta)\}}$$

The formulæ for the resistances R_1 and R_2 take very simple forms when we put $\alpha=0$ and $\mathrm{V}_1=0$.

We then have the solution for a cylinder moving in a viscous liquid bounded by an infinite rigid plane.

$$\mathrm{R}_2=-\frac{4\pi\mu\mathrm{V}_2\sinh 2\beta}{\{\beta\sinh 2\beta+1-\cosh 2\beta\}}$$

MOTION PARALLEL TO THE AXIS OF x.

5. Proceeding in an exactly similar way as in the former case the expression for the stream function is given by

$$\psi=(\mathrm{A}\cosh 2\eta+\mathrm{B}\sinh 2\eta+\mathrm{C}\eta)$$

$$+\left\{-\frac{1}{2}\mathrm{A}\cosh 3\eta-\frac{1}{2}\mathrm{B}\sinh 3\eta+\mathrm{E}\cosh\eta+\mathrm{F}\sinh\eta\right.$$

$$\left.+\mathrm{G}\eta\cosh\eta+\mathrm{H}\eta\sinh\eta\right\}\Big/(\cosh\eta+\cos\xi)$$

the absolute constant being omitted as it contributes nothing to velocity.

An expression similar to this was obtained by Jeffery in the paper cited in a different method, but the boundary condition being different the solution will be entirely different.

Let us suppose that the outer cylinder is moved with velocity U_1 and inner cylinder with velocity U_2 parallel to the axis of x

The boundary conditions are, when $\eta=\alpha$

$$\frac{\partial\psi}{\partial\xi}=-U_1 c\,\frac{\sinh\alpha\sin\xi}{(\cosh\alpha+\cos\zeta)^2}$$

$$\frac{\partial\psi}{\partial\eta}=-U_1 c\left\{\frac{\cosh\alpha}{(\cosh\alpha+\cos\xi)}-\frac{\sinh^2\alpha}{(\cosh\alpha+\cos\xi)^2}\right\}\qquad(e)$$

together with two similar conditions for the other cylinder where β and U_2 are written for α and U_1. But from the expression for the stream-function we get, when $\eta=\alpha$,

$$\frac{\partial\psi}{\partial\xi}=\Big\{-\frac{1}{2}A\cosh 3\alpha-\frac{1}{2}B\sinh 3\alpha+E\cosh\alpha+F\sinh\alpha$$

$$+G\alpha\cosh\alpha+H\alpha\sinh\alpha\Big\}\frac{\sin\xi}{(\cosh\alpha+\cos\xi)^2}$$

$$\frac{\partial\psi}{\partial\eta}=(2A\sinh 2\alpha+2B\cosh 2\alpha+C)$$

$$+\Big\{-\frac{3}{2}A\sinh 3\alpha-\frac{3}{2}B\cosh 3\alpha+E\sinh\alpha+F\cosh\alpha$$

$$+G(\cosh\alpha+\alpha\sinh\alpha)+H(\sinh\alpha+\alpha\cosh\alpha)\Big\}\Big/(\cosh\alpha+\cos\xi)$$

$$-\Big\{-\frac{1}{2}A\cosh 3\alpha-\frac{1}{2}B\sinh 3\alpha+E\cosh\alpha+F\sinh\alpha$$

$$+G\alpha\cosh\alpha+H\alpha\sinh\alpha\Big\}\frac{\sinh\alpha}{(\cosh\alpha+\cos\xi)^2}$$

The boundary conditions give the equations

$$-\frac{1}{2}A\cosh 3\alpha-\frac{1}{2}B\sinh 3\alpha+E\cosh\alpha+F\sinh\alpha$$

$$+G\alpha\cosh\alpha+H\alpha\sinh\alpha=-U_1c\sinh\alpha \quad \ldots \quad (8)$$

$$2A\sinh 2\alpha+2B\cosh 2\alpha+C \quad =0 \quad \ldots \quad (9)$$

$$-\frac{3}{2}A\sinh 3\alpha-\frac{3}{2}B\cosh 3\alpha+E\sinh\alpha+F\cosh\alpha$$

$$+G(\alpha\sinh\alpha+\cosh\alpha)+H(\alpha\cosh\alpha+\sinh\alpha)=-U_1c\cosh\alpha \ldots \quad (10)$$

together with three similar equations corresponding to the inner cylinder.

We have thus six equations but seven unknown quantities. But we know that ψ is a single-valued function. It follows therefore, that the velocities and consequently the pressure is a single-valued function. We can calculate p, the pressure, by noting that $\mu\nabla^2\psi$ and p are conjugate functions. In this way we find that p contains the many-valued term $2G\xi$, so that we must have

$$G=0 \quad \ldots \quad (11)$$

Now

$$c^2\nabla^2\psi=A\{1+4\cosh\eta\cos\xi+2\cosh 2\eta\cos 2\xi\}$$

$$+B\{2\sinh 2\eta\cos 2\xi+4\sinh\eta\cos\xi\}+2H\cosh\eta\cos\xi$$

$$+2E+2H$$

Therefore

$$p=-\frac{\mu}{c^2}[A\{2\sinh 2\eta\sin 2\xi+4\sinh\eta\sin\xi\}$$

$$+B\{2\cosh 2\eta\sin 2\xi+4\cosh\eta\sin\xi\}+2H\sinh\eta\sin\xi]$$

and

$$\gamma=c^{-2}\{L(\alpha)(\cosh\alpha+\cos\xi)+U_1c\sinh\alpha\cosh\alpha$$

$$+U_1c\sinh\alpha\cos\xi+4(A\cosh 2\alpha+B\sinh 2\alpha)(\cosh\alpha+\cos\xi)^2\}$$

The resistance R_1 can be calculated from the expression

$$R_1=\int\left(p\frac{dy}{ds}-U\frac{dx}{ds}\right)ds$$

where $U=\mu\gamma$ and the integration is taken round the circle.

$L(\alpha)$ in the expression for γ stands for the quantity

$$-\frac{9}{2}(A\cosh 3\alpha+B\sinh 3\alpha)+E\cosh\alpha+F\sinh\alpha$$

$$+H(2\cosh\alpha+\alpha\sinh\alpha)$$

$$R_1=-\frac{2\mu\pi}{c}\{L(\alpha)\cosh\alpha+U_1c\sinh\alpha\cosh\alpha+4(A\cosh 2\alpha+B\sinh 2\alpha)$$

$$+4\sinh\alpha(A\sinh 3\alpha+B\cosh 3\alpha)-2H\sinh^2\alpha\}$$

$=-\frac{4\mu\pi}{c}H$, after simplification.

$=4\mu\pi(U_1-U_2)X$

$$\frac{\{\cosh 2\alpha+\cosh 2\beta-4+4\cosh(2\alpha-2\beta)-\cosh(2\alpha-4\beta)-\cosh(4\alpha-2\beta)\}}{[(\beta-\alpha)\{\cosh(4\alpha-2\beta)+\cosh(2\alpha-4\beta)-4\cosh(2\alpha-2\beta)+4-\cosh 2\alpha-\cosh 2\beta\}+\{\sinh(4\alpha-2\beta)-\sinh(4\alpha-4\beta)+2\sinh 2(\alpha-\beta)-3\sinh 2\alpha+3\sinh 2\beta+\sinh(2\alpha-4\beta)\}]}$$

Similarly the force acting on the other cylinder is found to be

$$R_2=\frac{4\mu\pi}{c}H$$

When $\alpha=0$, $U_1=0$

$$R_1=\frac{4\mu\pi U_2}{\beta},$$

a very simple expression.

TRANSLATION OF TWO CYLINDERS IN AN INFINITE VISCOUS LIQUID.

6. In such a case, the velocity of the liquid does not vanish at infinity. To illustrate this point let us consider the motion parallel to the axis of y. The orthogonal components of velocity are

$$-h\frac{\partial\psi}{\partial\xi} \text{ and } h\frac{\partial\psi}{\partial\eta}.$$

At infinity $\eta=0$, $\xi=\pi$

$$-h\frac{\partial\psi}{\partial\xi}=-3c^{-1}(\text{E}+\text{H})$$

and

$$h\frac{\partial\psi}{\partial\eta}=-2c^{-1}(\text{E}+\text{H}).$$

Hence the motion being finite at infinity is inconsistent with the general supposition that the liquid is at rest at infinity. Hence the motion is impossible.

We should hardly wonder at this result. For Stokes has pointed out that the motion of a viscous liquid due to the translation of a circular cylinder never attains to a steady state, and our present problem is similar to that of Stokes.

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3

TRANSVERSE VIBRATIONS OF A THIN ROTATING ROD AND OF A ROTATING CIRCULAR RING.

BY

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I. INTRODUCTORY.

1. When a body rotates about an axis with constant angular velocity and is in relative equilibrium, every point of the body may be considered as being acted on by a force which varies as the distance of the point from the axis of rotation. The discussion of the vibrations of elastic solids acted on by such body-forces generally involves equations which cannot be solved in finite terms or in any convergent infinite series. It is probably due to this cause that very few problems relating to the vibration of rotating bodies have hitherto been solved. But an indirect method has often been applied in such cases to obtain the frequencies of vibrations which are very approximate for all practical purposes. This approximate method is due to Lord Rayleigh and one very interesting problem has been dealt with by Prof. Lamb and Mr. R. V. Southwell.[1] They have investigated the transverse vibrations of a thin homogeneous circular disc rotating about its axis with constant angular velocity. They observe that "the problem has a practical bearing, as throwing light on the occasional failure of turbine discs," which is most probably due to the transverse vibrations of these discs, causing the blades which are fitted to them, to come in contact with the adjacent parts of the machine. This problem of the laminar wheel suggests the case of a wheel with straight spokes and a circular rim, which is by no means a less common thing in mechanical contrivances.

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[1] "Vibrations of a Spinning Disc"—Proc. Roy. Soc., London. Ser. A., Vol. 99 (1921), pp. 272-280.

"On the Free Transverse Vibrations of Uniform Circular Disc clamped at its centre; and on the Effects of Rotation"—R. V. Southwell, Proc. Roy. Soc., London, Ser. A., Vol. 101 (1922), pp. 133-153.

2. It is clear that the discussion of the problem naturally resolves into two distinct parts, *viz.*, (1) the vibrations of the straight spokes, and (2) the vibrations of the circular rim. Both the spokes and the rim will be assumed to have small cross-sections, so that the effects of what is known as 'rotatory inertia' will be negligible. A spoke can vibrate transversally in two ways, either in the plane of the wheel or in a plane perpendicular to it. The mathematical solution is identical in the two cases. The rim may also vibrate in the same two ways; but the equations of motion are different, though it is known that the frequencies of the gravest modes of free vibration are very nearly the same.[1] When the spokes and the rim are taken as forming one body, the solutions become very complicated on account of the points of junction. In the work of the present paper, they are considered as separate bodies and independent solutions have been obtained for a thin rotating rod and a rotating circular ring.

II. Thin Rotating Rod.

3. Suppose that a rod (AB) of length a is rotating about A with constant angular velocity ω. Since the rod is thin, we assume the stress-system to consist of a longitudinal tension (T_x) only. If A be taken as origin and the axis of x along AB, we have

$$\frac{\partial T_x}{\partial x}+\rho\omega^2 x=0,$$

whence

$$T_x=\tfrac{1}{2}\rho\omega^2(A-x^2)$$

4. Case A. Let the end B be free, so that $T_x=0$ when $x=a$ and we have

$$T_x=\tfrac{1}{2}\rho\omega^2(a^2-x^2) \qquad \ldots \qquad (1)$$

Case B. Let a mass m [*e.g.* (mass of the rim)/(number of pokes)] be attached to B, so that when $x=a$, we have

$$T_x=m\omega^2 a.$$

Hence, in this case

$$T_x=\tfrac{1}{2}\rho\omega^2\left\{a\left(a+\frac{2m}{\rho}\right)-x^2\right\} \qquad \ldots \qquad (2)$$

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[1] Rayleigh Theory of Sound, Vol. I, Art. 192 a. Love, Elasticity, Chap. XXI, Art. 298.

5. Both the forms (1) and (2) may be included in the formula

$$T_x = \tfrac{1}{2}\rho\omega^2(c^2 - x^2) \quad \ldots \quad (3)$$

where

$$c^2 = a^2 \text{ or } a\left(a + \frac{2m}{\rho}\right),$$

according as the end B is free or carries a mass m.

When ω is very large and the flexural forces are negligible compared with the longitudinal tension, the equation of transverse vibration is

$$\rho\alpha dx \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left[T_x \cdot \alpha \frac{\partial v}{\partial x}\right] dx,$$

where α is the small cross-section, and v, the lateral displacement of an element of the bar at a distance x from the origin.

Substituting from (3) the value of T_x, we have

$$\frac{\partial^2 v}{\partial t^2} = \tfrac{1}{2}\omega^2 \frac{\partial}{\partial x}\left[(c^2 - x^2)\frac{\partial v}{\partial x}\right]$$

Assuming the solution

$$v = f(x)\cos(p_1 t + \epsilon)$$

we have

$$(c^2 - x^2)\frac{\partial^2 f}{\partial x^2} - 2x\frac{\partial f}{\partial x} + b^2 f = 0, \quad \ldots \quad (4)$$

where

$$b^2 = \frac{2p_1^2}{\omega^2} \quad \ldots \quad (5)$$

To solve this, it will be convenient to assume a series in ascending powers of $\frac{x}{c}$ a quantity which is never greater than unity. Let us assume

$$f(x) = A_0 + A_1\frac{x}{c} + A_2\left(\frac{x}{c}\right)^2 + \ldots + A_k\left(\frac{x}{c}\right)^k + \ldots \ldots \quad (6)$$

Substituting this in (4), we get

$$(c^2-x^2)\left[\ldots+\frac{k(k-1)}{c^k}A_k x^{k-2}+\ldots+\frac{(k+2)(k+1)}{c^{k+2}}A_{k+2}x^k+\ldots\right]$$

$$-2x\left[\ldots+\frac{k}{c^k}A_k x^{k-1}+\ldots\right]+b^2\left[\ldots+\frac{Ak}{c^k}x^k+\ldots\right]=0$$

Equating the coefficients of x^k to zero, we have

$$(k+2)(k+1)A_{k+2}=\{k(k+1)-b^2\}A^k.$$

Calculating the coefficients of (6) by this formula, we obtain

$$f(x)=A_0S_0(x)+A_1S_1(x),$$

where A_0 and A_1 are constants and $S_0(x)$ and $S_1(x)$ stand for the following series :

$$S_0(x)=1-\left[\frac{b^2}{2!}\left(\frac{x}{c}\right)^2+\frac{b^2(3.2-b^2)}{4!}\left(\frac{x}{c}\right)^4\right.$$

$$+\frac{b^2(3.2-b^2)(5.4-b^2)}{6!}\left(\frac{x}{c}\right)^6$$

$$\left.+\ldots+\frac{b^2(3.2-b^2)\ldots\{(2x-1)(2x-2)-b^2\}}{(2x)!}\left(\frac{x}{c}\right)^{2n}+\ldots\right]$$

$$S_1(x)=\frac{x}{c}+\frac{2\cdot1-b^2}{3!}\left(\frac{x}{c}\right)^3-\frac{(2\cdot1-b^2)(4\cdot3-b^2)}{5!}\left(\frac{x}{c}\right)^5+\ldots$$

$$+\frac{(2\cdot1-b^2)(4\cdot3-b^2)\ldots\{2x(2x-1)-b^2\}}{(2n+1)!}\left(\frac{x}{c}\right)^{2x+1}+\ldots$$

The complete solution is therefore

$$x=[A_0S_0(x)+A_1S_1(x)]\cos(p_1t+\epsilon) \qquad \ldots \qquad (7)$$

6. We have assumed the end A (*i.e.* $x=0$) to be fixed, so that we must have $v=0$ when $x=0$. This shews that we must put $A_0=0$, and the appropriate solution is

$$v=A_1S_1(x)\cos(p_1t+\epsilon) \qquad \ldots \qquad (8)$$

The series $S_1(x)$ is convergent when $x<c$ but it is divergent when $x=c$ or when $x>c$. We have now to distinguish between the two cases indicated in Art. 4 above.

In case A., we have $x=c(=a)$ at the edge, the series $S_1(x)$ is divergent and the solution is meaningless unless the series consists of a finite number of terms. Hence we see from the form of $S_1(x)$ that, in order that the series may terminate, b^2 must be of the form $2n(2n-1)$, where n is any positive integer. We therefore have

$$b^2 = 2n(2n-1)$$

or by (5),

$$p_1^2 = n(2n-1)\omega^2, \qquad \ldots \qquad (9)$$

n being any positive integer.

In case B., we have (from Art. 5)

$$c^2 = a\left(a + \frac{2m}{\rho}\right)$$

and x is always less than c. The series $S_1(x)$ is therefore always convergent. The condition of the end $x=a$, may be expressed by

$$\left[m\frac{\partial^2 v}{\partial t^2}\right]_{x=a} = \left[-\omega^2 a \cdot m\frac{\partial v}{\partial x}\right]_{x=a}$$

Substituting for v this becomes

$$p_1^2 S_1(a) - \frac{\omega^2 a}{c} S_1'(a) = 0,$$

or

$$b^2 S_1(a) - \frac{2a}{c} S_1'(a) = 0 \qquad \ldots \qquad (10)$$

which is an equation in p_1^2.

7. When, on the other hand, the influence of rotation is small compared with the flexural forces, we know that, the rotatory inertia of the cross-section of the rod being neglected, the equation of motion is

$$\frac{\partial^2 v}{\partial t^2} + \frac{Ek^2}{\rho}\frac{\partial^4 v}{\partial x^4} = 0$$

where k is the radius of gyration of the cross-section about a diameter perpendicular to the plane of vibration. If p_2 be the frequency, it is given by

$$p_2^2 = \frac{m^4 k^2}{4\pi^2 a^4}\,\frac{E}{\rho}. \qquad \ldots \qquad (11)$$

where m is given, in the case of a free-free bar, by $\cosh m \cos m = 1$. and in the case of a clamped-free bar, by $\cosh m \cos m = -1$.

8. When both the flexural and the centrifugal forces are taken into account, the equation of motion becomes

$$\frac{\partial^2 v}{\partial t^2}=\tfrac{1}{2}\omega^2\,\frac{\partial}{\partial x}\left[(c^2-x^2)\,\frac{\partial v}{\partial x}\right]-\frac{Ek^2}{\rho}\,\frac{\partial^4 v}{\partial x^4}.$$

If we assume

$$v=f(x)\cos(pt+\epsilon)$$

we have

$$\frac{2Ex^2}{\omega^2\rho}\,\frac{\partial^4 f}{\partial x^4}-(c^2-x^2)\,\frac{\partial^2 f}{\partial x^2}+2x\,\frac{\partial f}{\partial x}-b^2 f=0.$$

If a series analogous to (6) be substituted in this equation, the relation between the successive coefficients consists of three terms (*e.g.*, A_{k+4}, A_{k+2}, A_k) so that a general solution in finite terms or in a convergent infinite series is not easily obtainable.

We may, however, obtain approximate solutions by a method[1] indicated by Rayleigh.[2] According to this method, we may assume a given form for the displacement v, calculate the kinetic energy and equate this to the sum of the potential energies due to the angular motion and the flexural forces considered separately. The equation thus obtained yields the frequency of vibration.

We proceed to apply this method to the case A of Art. 4. The potential energy V of the centrifugal forces is given by

$$V=\tfrac{1}{2}\int aT_x\left(\frac{\partial v}{\partial x}\right)^2 dx,$$

where a=cross-section of the rod and

$$T_x=\tfrac{1}{2}\rho\omega^2(a^2-x^2)$$

The potential energy of the flexural forces is given by

$$V'=\tfrac{1}{2}\int Ek^2 a\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$$

The kinetic energy is given by

$$T=\tfrac{1}{2}\rho\int\left(\frac{\partial v}{\partial t}\right)^2 a\,dx.$$

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[1] This method has also been adopted by Prof. Lamb and Mr. R. V. Southwell in the papers cited.

[2] Theory of Sound, Vol. I, Chap. IV, Arts. 88 *et seq.*

Let us assume the form

$$v=f(x)\cos(pt+\epsilon).$$

Then

$$V=\tfrac{1}{4}\rho\omega^2 a\int(a^2-x^2)\left(\frac{\partial f}{\partial x}\right)^2\cos^2(pt+\epsilon)dx$$

$$V'=\tfrac{1}{2}Ek^2 a\int\left(\frac{\partial^2 f}{\partial x^2}\right)^2\cos^2(pt+\epsilon)dx$$

and

$$T=\tfrac{1}{2}\rho a\int p^2 f^2\sin^2(pt+\epsilon)dx$$

$$=p^2\cdot\tfrac{1}{2}\rho a\int f^2\sin^2(pt+\epsilon)dx$$

If V_f, V'_f and T_f denote the mean values of V and V', and the expression

$$\tfrac{1}{2}\rho a\int f^2\sin^2(pt+\epsilon)dx$$

respectively, we have

$$p^2=\frac{V_f+V'_f}{T_f} \qquad \dots \qquad (12)$$

The closer the assumed function $f(x)$ agrees with the actual form of the vibrating bar, the more will the value of p^2 approach

$$(V_f+V'_f)/T_f.$$

Moreover, the frequency remains stationary for small deviations from the actual type. Hence, if p_1 and p_2 be the two values of the frequency, obtained from the equation (9) or (10) and (11) respectively, we have [1] very approximately

$$p_1{}^2=\frac{V_f}{T_f}, \quad p_2{}^2=\frac{V'_f}{T_f}$$

and

$$p^2=p_1{}^2+p_2{}^2 \qquad \dots \qquad (13)$$

9. Assume as an example that

$$f(x)=A_1\left\{\frac{x}{a}+m\left(\frac{x}{a}\right)^3\right\} \quad \dots \quad (14)$$

where m is a variable parameter whose value is to be determined from the fact that the value of the period given by equation (12) should be a minimum.[1] Let us now calculate the values of V_f, V'_f and T_f. Since the mean values of

$\cos^2(pt+e)$ or $\sin^2(pt+\epsilon)$ is $\frac{1}{2}$, we find

$$V_f=\tfrac{1}{8}\rho\omega^2 aA_1\int_0^a(a^2-x^2)\left\{\frac{1}{a}+\frac{3m}{a^3}x^2\right\}^2dx$$

$$=\tfrac{1}{8}\frac{\rho\omega^2 aA_1}{a^6}\int_0^a\{a^6+(6m-1)a^4x^2$$

$$+(9m^2-6m)a^2x^4-9m^2x^6\}dx$$

$$=\frac{\rho\omega^2 aaA_1}{4\cdot105}(27m^2+42m+35)$$

$$V'_f=\tfrac{1}{4}Ek^2aA_1\int_0^a\left(\frac{6mx}{a^3}\right)^2dx$$

$$=\frac{3Ek^2aA_1}{a^3}m^2$$

$$T_f=\tfrac{1}{4}\rho aA_1\int_0^a\left\{\frac{x}{a}+m\left(\frac{x}{a}\right)^3\right\}^2dx$$

$$=\frac{\rho aaA_1}{4\cdot105}(15m^2+42m+35)$$

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[1] See remarks by Mr. R. V. Southwell in the paper "Vibrations of a Spinning Disc,"

As a partial verification of the above results, let us make $n=2$ in equation (9), so that

$$p_1^2=6\omega^2$$

The corresponding value of b^2 is 12, the series $S_1(x)$ terminates at the second term and

$$S_1(x)=A_1\left\{\frac{x}{a}-\frac{5}{3}\left(\frac{x}{a}\right)^3\right\},$$

so that

$$m=-\tfrac{5}{3}.$$

Substituting this value of m in the expressions for V_f and T_f found above, we see that

$$V_f=\frac{2A_1}{21}\rho\omega^2 a\alpha$$

and

$$T_f=\frac{A_1}{63}\rho a\alpha$$

so that

$$p_1^2=\frac{V_f}{T_f}=6\omega^2,$$

which is the same as that obtained from equation (9) by putting $n=2$.

To return to our general case, we have

$$V_f+V'_f=\frac{\rho\omega^2 a\alpha}{420}(27m^2+42m+35)+\frac{3Ek^2\alpha}{a^3}m^2$$

and

$$p^2=\frac{\dfrac{\rho\omega^2 a\alpha}{420}(27m^2+42m+35)+\dfrac{3Ek^2\alpha}{a^3}m^2}{\dfrac{\rho a\alpha}{420}(15m^2+42m+35)}$$

If for brevity, we put

$$\left.\begin{array}{lll} A=\frac{9}{140}\rho\omega^2 a+\dfrac{3Ek^2}{a^3} & & \\ B=\frac{1}{10}\rho\omega^2 a, & C=\frac{1}{12}\rho\omega\ \text{[illegible]} & \\ A'=\frac{1}{28}\rho a, & B'=\frac{1}{10}\rho a, & C'=\frac{1}{12}\rho a \end{array}\right\}, \quad \ldots \quad (15)$$

we get

$$p^2=\frac{Am^2+Bm+C}{A'm^2+B'm+C'}$$

We have now to find m in order that the values of p^2 may be stationary. The corresponding values of m are given by

$$(AB'-A'B)m^2-2(AC'-A'C)m+BC'-BC'=0 \quad \ldots \quad (16)$$

and the values of p^2 by

$$(A'C'-\tfrac{1}{4}B'^2)p^4-(C'A+A'C-\tfrac{1}{2}BB')p^2$$
$$+AC-\tfrac{1}{4}B^2=0 \quad \ldots \quad 17)$$

The values of m and p^2 may be calculated when the values of the constants (15) are known, and the true value of the frequency will be obtained, if the assumed form (14) is appropriate.

III. Rotating Circular Ring.

10. We assume that a circular ring of radius a and small cross-section, rotating in its plane with constant angular velocity ω, is vibrating transversally, the displacements being perpendicular to the plane of the ring. If p_1 and p_2 be the values of the frequency in the two extreme cases, *viz.*, (1) when the flexural forces are negligible and (2) when the angular motion is negligible, then, according to our observations in Art. 8, we have very approximately

$$p^2=p_1^2+p_2^2.$$

It is known[1] that, when the rotatory inertia is neglected, the value of p_2^2 is given by

$$p_2^2=\frac{E\pi c^4}{4ma^4}\,\frac{n^2(n^2-1)^2}{n^2+1+\sigma} \quad \ldots \quad (18)$$

where c is the radius of the cross-section, m the mass per unit length and n is any integer.

---

[1] Love, Elasticity, Art. 293 (b) or Michell, Messenger of Mathematics, XIX, 1889

We proceed to find p_1.

11. Taking the centre of the ring as origin and (a, θ) the polar co-ordinates of any point on the circumference, we have, assuming the stress-system to consist of a longitudinal tension only,

$$-\frac{1}{a}\widehat{\theta\theta}+\rho\omega^2 a=0,$$

whence

$$T_\theta(=\widehat{\theta\theta})=\rho\omega^2 a^2 \qquad \ldots \qquad (19)$$

The equation of motion is accordingly

$$\rho a\frac{\partial^2 v}{\partial t^2}a\,d\theta=\frac{\partial}{\partial\theta}\left[\,aT_\theta\frac{\partial v}{a\,\partial\theta}\right]d\theta$$

or

$$\frac{\partial^2 v}{\partial t^2}=\omega^2\frac{\partial^2 v}{\partial\theta^2} \qquad \ldots \qquad (20)$$

The solution of this equation is

$$v=A\cos(\mu\theta+\beta)\cos(p_1 t+\epsilon)$$

where

$$\mu^2=\frac{p_1^2}{\omega^2}.$$

(*i*) If the point $\theta=0$ of the ring is relatively fixed, we have

$$v=A\sin\mu\theta\cos(p_1 t+\epsilon).$$

Since, in this case, $v=0$ when $\theta=2s\pi$, s being any integer, we have

$$\sin 2\mu s\pi=0,$$

so that $2\mu=\frac{k}{s}$, k and s being any integers and

$$p_1=\frac{k}{2s}\omega \qquad \ldots \qquad (22)$$

(*ii*) If two diametrically opposite points, $\theta=0$ and $\theta=\pi$, are fixed, we must have

$$\sin \mu\pi=0$$

so that

$$\mu=s, \qquad \text{any integer,}$$

and

$$p_1=sw$$

(*iii*) If tha ends of a quadrant, $\theta=0$ and $\theta=\frac{\pi}{2}$, are fixed, we have

$$\sin\frac{\mu\pi}{2}=0$$

and

$$p_1=2s\omega,$$

where s is any integer.

(*iv*) *Generally*, if the ends of the arc, $\theta=0$ and $\theta=\frac{2\pi}{n}$, are fixed, then

$$\sin \mu\frac{2\pi}{n}=0$$

whence

$$p_1=\tfrac{1}{2}n\, s\, w,$$

s being any integer.

The solution (18) for p_2 refers to a complete ring. Hence the corresponding solution for p_1 may be taken from (22), and the period, when both the angular velocity and the flexural forces are taken into account, will then be given by the equation

$$p^2=p_1^2+p_2^2.$$

The results in (ii), (iii), (iv) give very simple relations between the angular velocities and periods of free transverse vibrations of thin flexible rotating arcs of any angle clamped at the extremities.

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4

GEOMETRICAL REPRESENTATION OF EQUATIONS OF CONICS FOR COMPLEX VARIABLES

BY

MANUJANATH GHATAK

Chapter I

§1. *The necessity for the introduction of four-dimensional space.*

The geometric representation of an analytic equation in x and y is ordinarily obtained by the admission of only real values for the variables x and y. The imaginary or complex values have no place there, and where two such equations have imaginary or complex solutions, we get no points in which the corresponding geometrical figures intersect. Hence we get the phenomenon in Conic Sections of a straight line and a conic sometimes intersecting and sometimes not intersecting, whereas the analytical equations always have solutions. In the latter case we say, to bring the geometrical phenomenon in line with analytical results, that the line intersects the conic in imaginary points. What is really the case is that there are no points common to the line and the conic.

The anomaly arises out of the fact that the roots of an equation with real coefficients give rise to numbers which are not always real. The Argand's diagram gives us a method of representing all these numbers in a plane, and the totality of these numbers covers up the entire two dimensional plane region. The real numbers as well as the purely imaginary numbers are but particular cases of complex numbers

Since for the adequate representation of a single complex variable X we require a plane or a space of two dimensions, the adequate representation of two complex variables X & Y would require two planes in a space of four dimensions, having a common point at the origin. The four coordinate axes will lie two and two in the two planes. In each plane there are an axis of reals and an axis of imaginaries which are at right angles to each other. All the coordinate axes may be at right angles to one another, but we may have sometimes to deal with oblique axes. It should be remembered, however, that the axis of imaginaries is necessarily at right angles to the axis of reals, but the angles between the axes in X-plane and axes in Y-plane will not always be right angles. In the case of equations to the Conic Sections, if we admit of complex values for the variables and substitute $\alpha+i\beta$ for x and $\gamma+i\delta$ for y, a single relation connecting x and y will be equivalent to two relations connecting α, β, γ and δ, which are obtained by equating the real and imaginary parts separately to zero. Hence we obtain that the equations really represent *surfaces*, whose sections in the plane of reals are what we ordinarily consider to be their geometrical interpretation. In reality, therefore, the equations to the straight line and the conic represent something more than the line or the conic in the real plane. They represent surfaces, and if the line and the conic in the real plane have no points of intersection, and still we can find solutions to the *equations*, we conclude that the *surfaces* intersect in some points outside the plane of reals.

As an illustration, let us take a case where the solution for one of the variables is purely imaginary, and see whether a three-dimensional space will give a geometrical solution.

Let the equations be $x^2+y^2=25$ and $x=\pm c$. When $|x|<5$ the solutions are real and the circle and the straight line in the real plane intersect in two points.

When $|x|=5$, the straight line touches the circle, and when $|x|>5$, there are no real solutions and the straight line and the circle do not meet.

In the real-imaginary plane, on the other hand, the curves are the hyperbola $x^2-y^2=25$, and the straight line $x=\pm c$; and when $|x|>5$, the straight line and the hyperbola intersect in two points; when $|x|=5$ they touch, the point of contact being the same as in the previous case, and is the common point of the circle and the hyperbola.

In each case, the points of intersection are those in which the *surface* $x^2+y^2=25$ intersects the *surface* $x=\pm c$, the point of contact being the point where $x=\pm5$ intersects $x^2+y^2=25$. Where the geometrical solutions were unavailable in the real plane, they were obtained in the real-imaginary plane.

Similar arguments apply with regard to the equations $x^2+y^2=25$ and $y=\pm c$, the alternative plane of solution being the imaginary-real plane. The equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x=\pm c$ or $y=\pm d$ are also of the same nature. The critical value in the case of $x=\pm c$ is $|x|=a$, and the alternative plane of solutions is the real-imaginary plane; in the case of $y=\pm d$ the critical value is $|y|=b$, and the alternative plane the imaginary-real plane. Similar considerations apply with regard to the equations $y^2=4ax$ and $x=c$, the manifestation of the surface in the real-imaginary plane is $y^2+4ax=0$ and the critical value is $x=0$.

In all these cases, we have deliberately chosen the equations in such a way that a three-dimensional space suffices to show all the points of intersection. This mode of representation has helped to show that the curve in the real plane is not the whole of the surface represented by the second degree equation, and that there are other planes where we also get curves of intersection. It is difficult, however, to get an accurate conception of a surface in four dimensions; we can study only its curve-sections, and imagine that the surface is made up of all these curves. We should remember, however, that not all planes give curves of section, and we shall have to choose our planes in such a way as to make this possible. We shall show later, that in the case at least of equations of first and second degree, a single infinity of planes may in all cases be obtained where we get curves of section, and that the totality of all these curves represents the entire surface.

§2. *The plane in four-dimensions.*

1. *The most general equations. Solid of the first degree.*

When the equation in four dimensions is of the first degree, we might call it *a solid of the first degree.*

The most general scheme of transformation of coordinates may be written,

$$\alpha'=a_1\alpha+b_1\beta+c_1\gamma+d_1\delta+e_1$$

$$\beta'=a_2\alpha+b_2\beta+c_2\gamma+d_2\delta+e_2$$

$$\gamma'=a_3\alpha+b_3\beta+c_3\gamma+d_3\delta+e_3$$

$$\delta'=a_4\alpha+b_4\beta+c_4\gamma+d_4\delta+e_4$$

and by its aid any equation of the first degree may be transformed into $\delta=0$. We might, therefore, get a conception of a solid of the first degree from the equation $\delta=0$ which embraces a three-dimensional Euclidean space. Any equation of the first degree would then be the analytical equivalent of the solid being given any desired positions in a four dimensional space.

We shall now prove that a plane in four-dimensional space is given by the intersection of any two equations of the first degree in four variables, The scheme of transformation given above would give a corresponding system of new axes. By this transformation any two equations may be reduced to the form $\beta'=0$, $\delta'=0$ which is a coordinate *plane* in the new system of axes. Hence the original equations must also represent this plane, and we get that any two equations of first degree in four variables represents a *plane*.

We shall now proceed with the problem of finding out this plane geometrically.

The general equations of a plane in four dimensions may be written,

$$\left.\begin{aligned} l\alpha+m\beta+n\gamma+p\delta&=a \\ l'\alpha+m'\beta+n'\gamma+p'\delta&=b. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (A)$$

By eliminating β and δ in succession between the two equations we can reduce them to the form,

$$\left.\begin{aligned} \beta&=a\alpha+c\gamma+e \qquad (i) \\ \delta&=b\alpha+d\gamma+f \qquad (ii) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (X).$$

A special advantage of writing the equations in this form is that a $(1-1)$ correspondence is established between the possible planes in a four-dimensional domain and the equations obtained by varying the constants. Such advantage does not belong to the equations (A) where all the variables are present in both. The same plane may, in that case, be represented by different pairs of equations.

Since with every change of the constants of the equations (X) a new plane is arrived at, and there are six of these constants, the number of planes possible in four dimensional space is six-fold infinity.

Turning now to the equations we see that (*i*) represents a plane in three-dimensional geometry (this is really the intersection of the two solids of the first degree $\beta=a\alpha+c\gamma+e$ and $\delta=0$). Thus the solid (*i*) passes through the plane $\beta=a\alpha+c\gamma+e$ in (α, β, γ) space. Again (*ii*) is a solid whose section by $\delta=0$ is the plane in three dimensions $b\alpha+d\gamma+f=0$. Now in (α, β, γ) space $\beta=a\,\alpha+c\,\gamma+e$ and $b\,\alpha+d\,\gamma+f=0$ together represent a *straight line*, which being common to the two planes is common to the two solids (*i*) and (*ii*). Hence the plane (X) has this line lying on it.

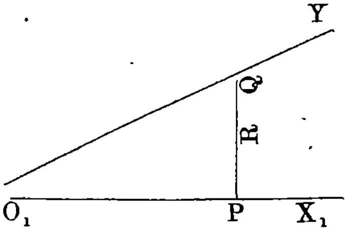
So much for the (α, β, γ) space. In (α, γ, δ) space, similarly, we get the planes of section to be $a\,\alpha+c\,\gamma+e=0$ and $\delta=b\alpha+d\,\gamma+f$ and these determine by their intersection a *straight line* which is common to (*i*) and (*ii*) and, therefore, to the plane (X).

Now these straight lines have the point given by $a\,\alpha+c\,\gamma+e=0$ and $b\,\alpha+d\,\gamma+f=0$ in the plane of reals, common. Hence these are coplanar and as the surface of intersection of two solids of first degree has been shown to be a plane, the equations (X) are the analytical equivalent of the plane defined by these two straight lines.

We might, however, show that every point in the plane determined by these two straight lines lies on both the solids and, therefore, on their surface of intersection, and thus get an alternative proof of the fact that the intersection of two solids of first degree is a plane.

To show that the plane determined by the line $b\,\alpha+d\,\gamma+f=0$ *and* $\beta=a\,\alpha+c\gamma+e$ *in* (α,β,γ) *space, and the line* $a\,\alpha+c\,\gamma+e=0$ *and* $\delta=b\,\alpha+d\,\gamma+f$ *in the* (α,γ,δ) *space, is the surface of intersection of (i) & (ii).*

Let O_1X_1 and O_1Y_1 be the straight lines. Then if $(\alpha', \beta', \gamma', 0)$ be any point P on O_1X_1 and $(\alpha'', 0, \gamma'', \delta'')$ any point Q on O_1Y_1 any point, R on PQ will be given by $(\alpha'+k\alpha'', \beta', \gamma'+k\gamma'', k\delta'')$.



But if P & Q lie on both the solids, the point R will also lie on them, and hence on their surface of intersection. But by varying the points on the lines O_1X_1 and O_1Y_1 and k, we can make R coincide with any point in the plane; hence the *plane* $X_1O_1Y_1$ lies altogether on the surface of intersection of (i) & (ii); or the two coincide.

§3. *The planes of examination*

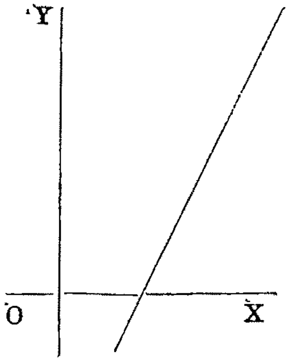
We now pass on to notice some of the most important particular cases of planes in four dimensions.

Where one of the two equations-defining the plane contains two of the variables (α and γ), and the other, the other two (β and δ) we get what may be termed *a plane of examination.*

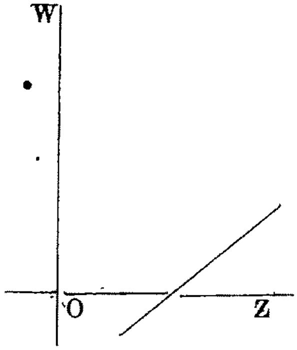
The equations might be written

$$\left.\begin{aligned}\gamma&=m\alpha+c\ldots(iii)\\ \delta&=m'\beta+d\ldots(iv)\end{aligned}\right\}\qquad\ldots\qquad\ldots\quad (Y)$$

We now proceed to a geometrical consideration of these equations in defining the plane formed by them, Let us take two planes, the real and the imaginary, and draw in them the straight lines



Plane of reals



Plane of imaginaries

which have the given equations (*iii*) & (*iv*) and which are really the sections of the solids (*iii*) and (*iv*) by these planes.

Definitions

The real and the imaginary plane together may be called the *basic planes.*

The complex point $(\alpha+i\beta, \gamma+i\delta)$ determined by the points (α, γ) and (β, δ) in the real and imaginary planes is said to be *formed by their association.* The points in the basic planes may be termed its *components.*

The *plane formed by the association of a line in the real plane and a line in the imaginary plane* is that defined by lines drawn parallel to them through the complex point formed by the association of a point on the real line of association, and a point on the imaginary line of association,

These lines of association are, of course, at right angles.

Analytically, the equations to this plane are given by (*iii*) and (*iv*) together, for these two together represent a plane. And the (α, γ) coordintes of the plane satisfy (*iii*) and the (β, δ) cordinates satisfy (*iv*). Hence both (*iii*) and (*iv*) pass through this plane, which therefore must coincide with their plane of intersection.

Hence we obtain, that there is only one plane formed by the association of a line in the real plane and a line in the imaginary plane, We now prove some important propositions with regard to these planes of examination. For convenience the word "plane" in what follows will mean a "plane-of examination."

Proposition I

Two planes will interest in the complex point formed by the association of the points of intersection, in the basic planes, of the lines of association of the planes.

This follows from the definition of the planes. If B & D be the points of intersection of the lines of association, the complex point (B, D) formed by the association of B & D, lies on both the planes and is their point of intersection.

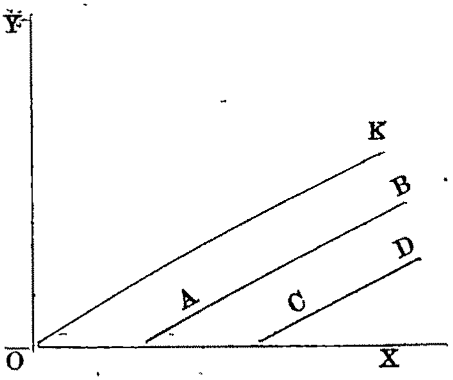
Hence it follows that two planes will, in general, intersect in only a single point; for the lines in the basic planes intersect in only a single point unless coinciding.

Proposition II

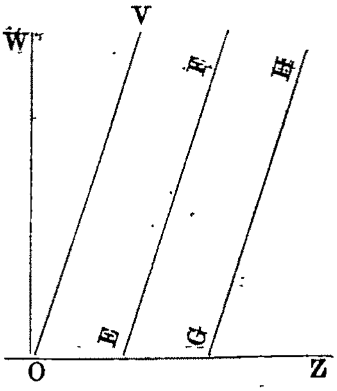
Two planes are parallel when they are formed by the association of lines which are parallel straight lines in their planes of reference.

Definition of parallelism.

Two planes are parallel when the line at infinity of one coincides with the line at infinity of the other.

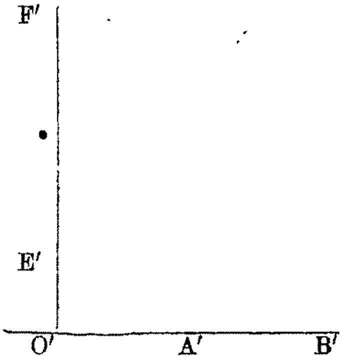


Plane of reals.



Plane of imaginaries.

Let (AB, EF) and CD, GH) be the two planes and AB be parallel to CD and EF to GH. By Prop. I the point of intersection is the complex point formed by the association of the points at infinity along AB & EF.



If we draw the plane of examination (AB, EF) having as origin a point O′ formed by the association of a point on AB and a point on EF, the complex point of intersection will be that determined by a point at infinity along A′B′ and a point at infinity along E′F′ (these being lines parallel to AB and EF through O′). But these coordinates do not define a single unique point but an infinity of points infinitely distant and lying on the line at infinity. Hence the points of intersection lie on the line at infinity on (AB, EF). Similarly they lie on the line at infinity on the plane (CD, GH). Thus the two planes have identical lines at infinity *i.e*, are parallel.

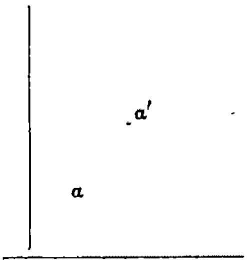
Proposition III

Two planes are also parallel when one pair of parallel lines of association become coincident.

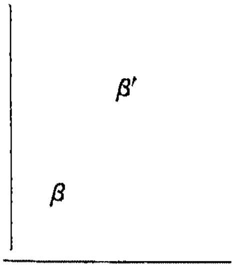
This follows from the preceding proposition when we remember that the line at infinity includes points, one of whose coordinates is finite. [These, of course, are one or other of the two points at infinity lying on the axes].

Proposition IV

Of the two systems of doubly infinite planes which pass through two different points, among corresponding parallel planes there is only a single pair which is coincident.



Plane of reals



Plane of imaginaries.

Let (α, β) and (α', β') be the points through which the systems of planes are drawn. The corresponding parallel planes are those which have their axes parallel (*i.e.*, are formed by the association of lines which are parallel in their basic planes). In the case in which the parallel axes get coincident (*i.e.*, in the case of the plane ($\alpha\alpha'$, $\beta\beta'$) we have coincident planes. The plane of examination ($\alpha\alpha'$, $\beta\beta'$) is common to both the systems.

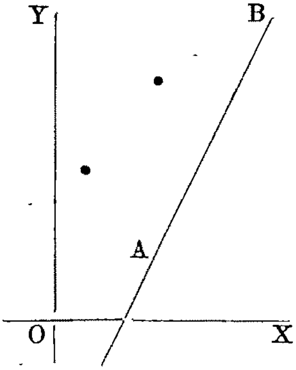
Proposition V

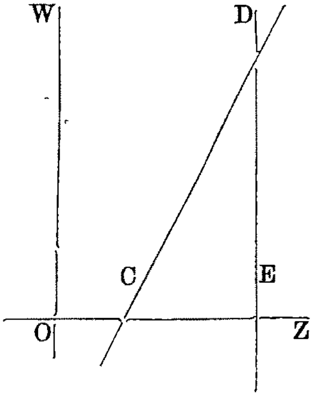
Two planes will meet in a single point at infinity when one of the lines of association of one is parallel to that of the other.

Prop. I gives us that the point of intersection is that formed by the association of the finite points of intersection in one basic plane, and a point at infinity in the other. When we draw one of the planes we find that the point of intersection is at infinity along an axis.

Proposition VI

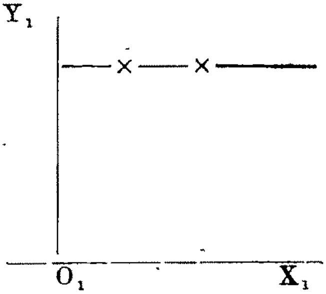
If a line of association of one plane coincides with that of the other, the two planes intersect in a line parallel to the coincident line of association.





Let the coincident line of association be in the real plane and let it be AB, and let CD and DE be the imaginary lines of association intersecting at D. Then Prop. I gives the points of intersection as those formed by association of every point of AB with D.

Let us draw the plane (AB, CD), and let it have as origin O_1 found in the usual way. Then in this plane the points of intersection will have the same y-coordinate, and, therefore the line of intersection will be parallel to O_1X_1 and hence to the coincident line of association AB.



The analytical proof is also interesting. Let the planes be

$$\left.\begin{array}{l}\gamma=m\alpha+c,\\ \delta=m_1\beta+d_1,\end{array}\right\} \qquad \left.\begin{array}{l}\gamma=m\alpha+c,\\ \delta=m_2\beta+d_2.\end{array}\right\}$$

The equations to the common line of intersection are,

$$\left.\begin{array}{l}\gamma=m\alpha+c,\\ \delta=m_1\beta+d_1,\\ \delta=m_2\beta+d_2.\end{array}\right\}$$

And the straight line is obviously one parallel to the line $\gamma=m\alpha+c$ in the real plane, through the point $(0, \beta, 0, \delta)$ where β and δ are determined from the last two equations

§4. *The plane of the first degree*

We now come to another particular case oft he general equations to a plane. It is furnished by the general equation of the first degree in two variables.

The most general form of the equation of the first degree is $y=mx+c$.

Splitting up the real and imaginary parts after substituting $\alpha+i\beta$ for x and $\gamma+i\delta$ for y,

$$\gamma=m\alpha+c\ldots(a) \qquad\qquad \delta=m\beta\ldots(b).$$

This shows that the equation whose manifestation is a straight line in the plane of reals is, in reality, a plane. We see also that it belongs to the class of the planes of examination. Such a plane is termed *a plane of the first degree.*

Planes of first degree are, however, particular cases of the planes of examination.

For, from the equations (a) & (b) we see, (a), that the lines of association are inclined at the same angles to the coordinate axes in their respective basic planes, and (β) that the line of association in the imaginary plane passes through the origin. These conditions doubly limit the possible number of planes and we get the totality of such planes to be only a two-fold infinity. The single equation in two variables to the surface also shows this to be the case.

The two following propositions with regard to planes of first degree are of importance.

Proposition A

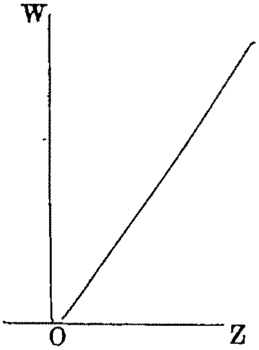
Equations of first degree in two variables which represent parallel lines in the plane of reals are really parallel planes meeting in a single infinity of points, situated altogether at infinity.

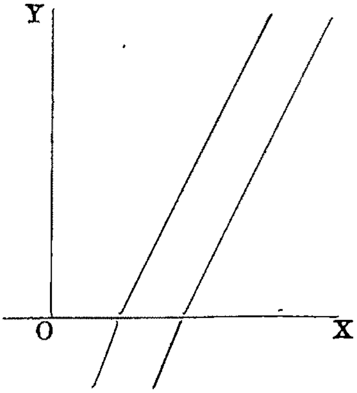
Let us take the two equations

$$y=mx+c\ldots(c)$$

and

$$y=mx+c'\ldots(d)$$





For equation (c) the lines of association are,

$$\left.\begin{array}{l}\gamma=m\alpha+c,\\ \delta=m\beta.\end{array}\right\}\ldots$$

For equation (d) they are,

$$\left.\begin{array}{l}\gamma=m\alpha+c'\\ \delta=m\beta\end{array}\right\}\ldots$$

Since the planes have parallel lines of association in the real plane, and coincident lines of association in the imaginary plane they represent parallel planes by Prop. III p. 180.

The second part of the proposition is proved from the definition of parallel planes.

A complementary proposition with regard to all other equations of first degree is given by,

Proposition B

In all other cases they represent planes having only a single point of intersection, which is in the real plane.

This follows from the fact that two first degree equations in two variables can only have a real solution. The proposition might also be proved by the theory of planes of examination by the help Prop. I p. 179. The lines in the imaginary plane pass through the origin; the lines in the real plane intersect at a definite point. Hence the point of intersection is obtained by associating the definite point in the plane of reals with the origin in the plane of imaginaries. Hence the point lies in the real plane, and is the intersection of the real lines of association.

The plane of the first degree with complex coefficients

We shall now consider the first degree equation in two variables, where the constants are complex quantities, and see what the equation represents under these circumstances.

Let the equation be

$$y=(A+iB)x+C+iD,$$

or

$$\gamma+i\delta=(A+iB)(\alpha+i\beta)+C+iD.$$

Splitting up real and imaginary parts,

$$\gamma=A\alpha-B\beta+C,$$

$$\delta=B\alpha+A\beta+D.$$

Hence the plane belongs to the most general class though it is a special case and contains only four constants.

The section of this plane by the solid $\delta=0$, is the straight line,

$$\left.\begin{aligned}\gamma&=A\alpha-B\beta+C,\\ 0&=B\alpha+A\beta+D,\end{aligned}\right\}\cdots$$

in the (α, β, γ) space.

The direction-cosines of the line are proportional to

$$A, -B, A^2+B^2.$$

Since every line in the (α, β, γ) space is perpendicular to the δ-axis, the fourth direction cosine of the line is 0.

Hence the direction-cosines are proportional to,

$$A, -B, A^2+B^2, 0$$

Similarly the section of the plane by $\gamma=0$ is the line

$$\left.\begin{aligned} 0&=A\alpha-B\beta+C,\\ \delta&=B\alpha+A\beta+D, \end{aligned}\right\}\ldots$$

in the (α, β, δ) space.

The direction-cosines are similarly proportional to,

$$B, A, 0, A^2+B^2.$$

Hence applying $ll'+mm'+nn'+pp'=0$ we see that the two characteristic lines are at right angles to each other.

Hence, the same method as in the case of the most general equation gives the geometrical location of the plane. In this case the characteristic lines determining the plane are found to be at right angles. This will be of use in determining curve-sections of surfaces and solids in such a plane.

§5. *Applications.*

The ground having been thus prepared, we shall now deal with the problem of the intersections of equations in two variables, where the solutions are not available in the real plane.

Problem I.

To find the points or lines of intersection of the surfaces given by $x^2+y^2=0$ and $x^2+y^2-a^2=0$

Put

$$x=\alpha+i\beta,\ y=\gamma+i\delta, \text{ and the equations become,}$$

$$(\alpha+i\beta)^2+(\gamma+i\delta)^2=0. \quad (\alpha+i\beta)^2+(\gamma+i\delta)^2=a^2.$$

whence we get

$$\left.\begin{aligned} \alpha^2-\beta^2+\gamma^2-\delta^2&=0\ldots(1)\\ \alpha\beta+\gamma\delta&=0\ldots(2). \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha^2-\beta^2+\gamma^2-\delta^2&=a^2\ldots(3).\\ \alpha\beta+\gamma\delta&=0\ldots(4). \end{aligned}\right\}$$

Equations (2) and (4) are identical, and we have a case of curve-intersection of the two surfaces.

(1) & (3) are inconsistent together unless $\alpha^2+\gamma^2$ and $\beta^2+\delta^2$ tend to become infinite, approaching each other in a ratio of equality.

From (2) we have,

$$\frac{\alpha}{\gamma}=\frac{-\delta}{\beta}=k \text{ suppose}\ldots(5).$$

This shows that $\alpha, \beta, \gamma, \delta$ must all be infinite for the points of intersection. From (5) we find that the points at infinity where the curves intersect are of the form $(a+i\beta,\ ia-\beta)$ or $(X,\ iX)$ where X gets infinite along a particular radius vector given by the ratio of α and β. Since, however, this ratio is indefinite, we have the case of a single infinity of points at infinity. Hence the surfaces intersect in a *curve* at infinity. We shall now determine whether these *touch* at infinity all along the curve. The sections in the real-imaginary and imaginary-real plane seem to suggest that this might be the case.

Before doing so, however, let us develop a method of obtaining an infinity of planes where we may get curve-sections of the surfaces. Let us analyse the equations (1) & (2) viz,

$$\alpha^2-\beta^2+\gamma^2-\delta^2=0\ldots(1)$$

$$\alpha\beta+\gamma\delta=0\ldots(2)$$

(2) by itself denotes a surface, and the planes obtained by giving to k all real values in,

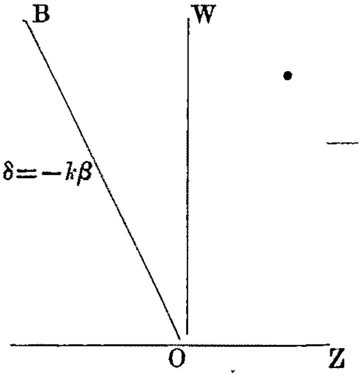
$$\frac{\alpha}{\gamma}=\frac{-\delta}{\beta}=k\ldots(5).$$

lie entirely on the surface. Hence the surface consists entirely of these planes, (which, it will be seen, are planes of examination). This is a case analogous to the generating lines of a ruled surface. They may be termed the *generating planes* of the solid.

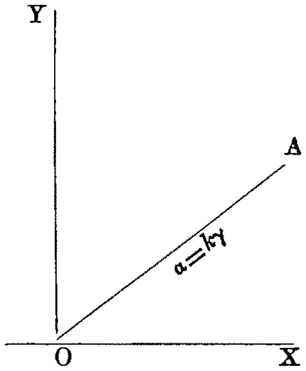
Thus the intersection of these planes with equation (1) will give the entire surface. The curves in these, in their totality, represent the whole surface.

Now for a definite value of k, the equations (5) represent two equations. These two equations combined with (1) give us three equations among the four variables. We have thus for their intersection a certain curve, and we proceed te analyse this curve and see what it geometrically represents.

In order to determine the section in the plane $\alpha=k\gamma$ $\delta=-k\beta$, for any value of k the following artifice may be adopted.



Plane of imaginaries.



Plane of reals.

The plane is formed by the association of the lines OA & OB in the real and imaginary planes whose equations are $\alpha=k\gamma$, $\delta=-k\beta$.

Take for axes the lines OA & OB which are at right angles to each other. A unit length along OA will have the cordinates,

$$\alpha=\frac{k}{\sqrt{1+k^2}},\ \gamma=\frac{1}{\sqrt{1+k^2}}$$

and a unit length along OB,

$$\beta=\frac{1}{\sqrt{1+k^2}},\ \delta=\frac{-k}{\sqrt{1+k^2}}.$$

In order to determine the curve in the plane our method will be to take any point (m, m') referred to these lines as axes, and then to express the original coordinates in terms of m & m'. The substitution of these in the given equation will give us a relation between m & m' which will be the curve required in the plane.

The point (m, m') in this plane has for its coordinates,

$$\alpha=\frac{mk}{\sqrt{1+k^2}},\ \gamma=\frac{m}{\sqrt{1+k^2}}$$

$$\beta=\frac{m'}{\sqrt{1+k^2}},\ \delta=\frac{-m'k}{\sqrt{1+k^2}}.$$

and substituting these in equation (1) viz.

$$\alpha^2-\beta^2+\gamma^2-\delta^2=0.$$

we get

$$\frac{m^2k^2}{1+k^2}-\frac{m'^2}{1+k^2}+\frac{m^2}{1+k^2}-\frac{m'^2k^2}{1+k^2}=0.$$

or

$$m^2-m'^2=0.$$

[We would have got the same result if we had substituted these values in $x^2+y^2=0$, where $x=\alpha+i\beta$, $y=\gamma+i\delta$].

The equation obtained is independent of k and we see that in all these planes the section is the same; viz the pair of straight lines $m^2-m'^2=0$.

In the case of the equation $x^2+y^2=a^2$, we see in an exactly similar way that the section is the rectangular hyperbola, $m^2-m'^2=a^2$.

Since the planes of examination in both these cases are identical, the two surfaces touch each other at infinity at two points in each of these planes, where the m & m' coordinates are in the ratio of equality, and, therefore, the x & y coordinates are of the form (X,iX), or (iX,X). This happens in all the planes obtained by varying k, and as these planes, in their totality, contain the entire surface defined by the two equations, these touch each other all along the single infinity of points thus obtained; *i.e.*, they touch each other all along their curve of intersection.

In the case of the equation $x^2+y^2=-a^2$ the same process will give the section in the identical planes of examination to be $m^2-m'^2+a^2=0$ and this represents the conjugate rectangular hyperbola, and the surface touches in the curve at infinity the other two surfaces.

The same is true of all the equations of the form $x^2+y^2=a^2$ obtained by varying a^2, which are concentric circles in the plane of reals and concentric rectangular hyperbolas in the single infinity of planes of examination $\alpha=k\gamma$, $\delta=-k\beta$. And we deduce that they all touch in their common curve at infinity at every point of which the x & y coordinates are in the ratio of $1:i$.

Problem II.

To determine the points and lines of intersection of the surfaces given by $x^2+y^2=a^2$, *and* $(x-h)^2+y^2=b^2$.

The equations become, when broken up into real and imaginary parts,

$$\alpha^2+\gamma^2-\beta^2-\delta^2=a^2 \text{---}(1') \qquad (\alpha-h)^2+\gamma^2-\beta^2-\delta^2=b^2 \text{---}(3')$$

$$\alpha\beta+\gamma\delta=0 \text{---}(2') \qquad (\alpha-h)\beta+\gamma\delta=0 \text{---}(4')$$

The finite points of intersection might be obtained by solving the equations. For the points of intersection at infinity we shall adopt the analysis of the preceding example.

As before, the sections in the planes, $\frac{\alpha}{\gamma}=\frac{-\delta}{\beta}=k$, in the case of the equations (1′) & (2′) and those in the planes $\frac{\alpha-h}{\gamma}=\frac{-\delta}{\beta}=k$ in the case of the equations (3′) & (4′), are rectangular hyperboles whose asymptotes are $m^2-m'^2=0$.

Now the planes $\alpha=k\gamma$, $\delta=-k\beta$ and $(\alpha-h)=k\gamma$, $\delta=-k\beta$ are parallel (by Prop. III p. 180). They intersect in the line at infinity in their planes.

[That the planes intersect in a line is also apparent from the fact that their equations are equivalent to the following three equations:— $\alpha=k\gamma$, $\alpha-h=k\gamma$ and $\delta=-k\beta$.]

The parallel asymptotes in the parallel planes, and therefore also the rectangular hyperbolas intersect in points at infinity which lie on this line at infinity. This happens in the case of the single infinity of planes obtained by giving k all real values. Thus we obtain that the surfaces intersect in two finite points and a single infinity of points at infinity, or a curve at infinity. The coordinates of the points at infinity along the system of planes $\alpha=k\gamma$, $\delta=-k\beta$ are $\{A+iB, i(A+iB)\}$ where A & B have infinite values in any ratio: *i. e.* $A+iB$ may become infinite along any vector.

[We also see why there should be a curve of intersection of the surfaces, from a consideration of the equations. For the four equations are really equivalent to the three:—$\alpha=\infty$, $\alpha^2-\beta^2+\gamma^2-\delta^2=0$ and $\alpha\beta+\gamma\delta=0$, a curve altogether at infinity].

The problem is very similar in the case of the equations

$$a^2+\gamma^2-\beta^2-\delta^2=a^2 \text{ — } (1'') \qquad (a-h)^2+(\gamma-k)^2-\beta^2-\delta^2=b^2 \text{ — } (3'')$$

$$a\beta+\gamma\delta=0 \text{ — } (2'') \qquad (a-h)\beta+(\gamma-k)\delta=0 \text{ — } (4'')$$

In the case of equations (1″) and (2″) the single infinity of planes $\frac{a}{\gamma}=\frac{-\delta}{\beta}=k$ have curve-sections of the surface which are rectangular hyperbolas. The corresponding planes are parallel by Prop III p. 180 and the asymptotes being parallel lines in these planes intersect in two points at infinity on the line at infinity on both. The curve-sections, consequently, intersect and the surfaces have points of intersection in each of the single infinity of planes obtained by giving k all real values. Regarding the finite points of intersection the ordinary methods suffice. Combining all these we get,

(*i*) The equations which in the plane of reals are concentric circles, are surfaces which touch at all points on a curve at infinity whose x, y coordinates are in the ratio $1:\pm i$. These points have coordinates of the form $[A+iB, \pm i(A+iB)]$ where A & B are infinite and different points are obtained by varying the ratio in which they become infinite.

(ii) All equations representing circles in the plane of reals have a common curve of intersection which is the circle at infinity. Any two of these have, besides two finite points of intersection.

And now we can see why it is that *two circles can never intersect in more than two points,* whereas two conics will generally have four points of intersection. The corresponding algebraical equations with which the circles have been associated have two finite and two infinite solutions, and the points corresponding to the infinite solutions are always outside the plane of reals. The finite solutions give rise to finite points and where these are real the circles intersect. But the infinite solutions give rise to the curve at infinity whose x & y coordinates are in the ratio of $1:\pm i$, and which is, therefore, absolutely outside the plane of reals.

Two concentric circles can never intersect.—For the corresponding algebraical equations with which they are associated have two pairs of coincident infinite solutions, giving rise to the circle at infinity twice. Hence these can have no finite points of intersection. The infinite points are outside the plane of reals.

The following examples illustrate the method by which we can bring to view the surface represented by the general equation of the second degree in two variables, the consideration of which will appear in the next Chapter.

Problem III.

To determine a single infinity of planes which intersect the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in curves, and to find the equations to the curve-sections in these planes.

The first portion is solved by a method similar to the preceding. On separation of real and imaginary parts, the equation splits up into,

$$\left.\begin{array}{l}\frac{\alpha^2-\beta^2}{a^2}+\frac{\gamma^2-\delta^2}{b^2}=1\\[2ex] \frac{\alpha\beta}{a^2}+\frac{\gamma\delta}{b^2}=0\end{array}\right\}\text{—(A).}$$

From the second equation we have our system of planes to be,

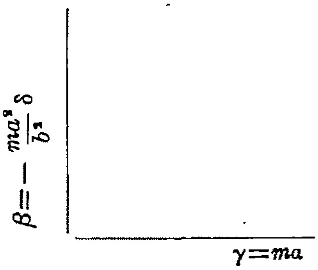
$$\frac{\gamma}{\alpha}=-\frac{b^2}{a^2}\cdot\frac{\beta}{\delta}=m.$$

or

$$\gamma=m\alpha,\ \beta=-\frac{ma^2}{b^2}\,\delta.$$

To find the equation in this plane we take the lines of association $\gamma=m\alpha$ in the real plane, and $\beta=-\frac{ma^2}{b^2}\,\delta$ in the imaginary plane as our axes.

If (k, k') be any point in the plane with reference to these axes, the $(\alpha, \beta, \gamma, \delta)$ coordinates of the point are,



$$\alpha = \frac{k}{\sqrt{1+m^2}}, \gamma = \frac{mk}{\sqrt{1+m^2}}$$

$$\beta = -\frac{ma^2k'}{\sqrt{b^4+m^2a^4}}, \delta = \frac{b^2k'}{\sqrt{b^4+m^2a^4}}.$$

On substitution of these values in the first of equations (A) we get the locus to be,

$$\frac{k^2}{a^2b^2(1+m^2)} - \frac{k'^2}{b^4+m^2a^4} = \frac{1}{m^2a^2+b^2}$$

We thus see that the sections are different in different planes. The aggregate of all these curves is the surface itself. Its manifestation in the plane of reals is what we ordinarily associate with the representation of the equation. We see, however, that all the curves are conic sections.

We take another problem to illustrate the case where the lines of association do not pass through the origin.

Problem IV.

To solve a similar problem in the case of the equation $y^2 = 4ax$.

The equation may be written,

$$(\gamma + i\delta)^2 = 4a(\alpha + i\beta)$$

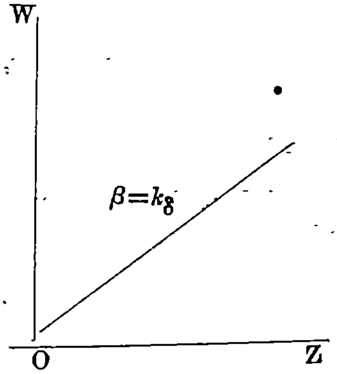
which splits up into,

$$\left.\begin{aligned} \gamma^2 - \delta^2 &= 4a\alpha \\ \gamma\delta &= 2a\beta \end{aligned}\right\} \dots (A').$$

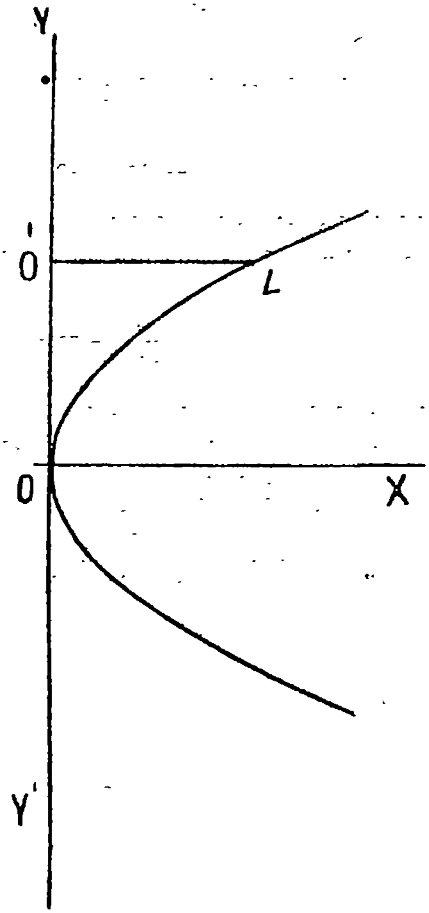
From the latter we get the equations of the planes of examination as,

$$\frac{\gamma}{2a} = \frac{\beta}{\delta} = k$$

and the lines of association in the real and imaginary plane are, $\gamma=2ak$, $\beta=k\delta$.



Plane of imaginaries.



Plane of reals,

Let O_1 be the point where the real line of association intersects OY; and (OO_1) (*i.e.*, formed by association of O & O_1), be the origin of coordinates in our plane of examination.

The point (m, m') will have its coordinates,

$$\alpha=m,\ \gamma=2ak$$

$$\beta=\frac{km'}{\sqrt{1+k^2}},\ \delta=\frac{m'}{\sqrt{1+k^2}}.$$

Substituting in the first of equations (A′), the equation to the complementary curve becomes,

$$4a^2k^2-\frac{m'^2}{1+k^2}=4am$$

or

$$\frac{m'^2}{1+k^2} = -4a(m-ak^2)$$

Transferring the origin to the point $(ak^2, 0)$ with reference to new axes, or to

$$\alpha=ak^2,\ \beta=0,\ \gamma=2ak,\ \delta=0,$$

which is the point where O′L intersects the parabola in the real plane, the equation to the complementary parabola becomes,

$$\frac{m'^2}{1+k^2} = -4am.$$

This parabola meets the parabola in the real plane, and has its axis in the opposite direction. All points along O′L where m is positive are within the principal curve, and for negative values of m the points are within the curve in the plane of examination.

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5

THE INVESTIGATIONS OF THE FORCED OSCILLATIONS SET UP IN AN AEROPLANE BY PERIODIC GUSTS OF WIND, WITH SPECIAL REFERENCE TO THE CASE OF SYNCHRONY WITH THE FREE OSCILLATIONS.

BY

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The " Phugoid theory " propounded by Lanchester and elaborated by Bryan furnishes us with the path and the periods of the natural oscillations of an aeroplane which has been plunged with a velocity slightly different from its natural velocity. The equation of the path given by Lanchester for an aerodone is

$$\text{Cos } \theta = \frac{H}{3H_*} + \frac{C}{\sqrt{H}}$$

when θ is the angle of path to horizon, H_* the natural height i.e., the height of fall corresponding to the actual velocity and C a variable parameter. For different value of C Lanchester has plotted the flight-path in his " Aerial Flight " and discussed their various possibilities, From the form of the equation it is evident that the path is a periodic one and Bryan in his Stability in Aviation proceeding in a more mathematical and extended way finds that the periods of the natural oscillations are given by the roots of a bi-quadratic which he has solved for particular cases. In general he finds that the motion consists of 2 distinct periodic oscillations, the time period of one is very long and another comparatively short. In fact Thompson solving a particular case of an aeroplane with the natural velocity 100 ft. per sc. finds that the roots of Bryan's bi-quadratic are given by

$$\lambda^4 + 5 \cdot 2\lambda^3 + 11 \cdot 6\lambda^2 + 1 \cdot 38\lambda + + 1 \cdot 04 = 0$$

so that the roots are

$$\lambda = -2 \cdot 64 \pm 2 \cdot 14 \sqrt{-1}$$

$$\lambda = - \cdot 089 \pm \cdot 297 \sqrt{-1}$$

It is worth noticing that the first pair of roots are about seven times the second pair, while the damping factor in the former case is about 70 times that of the second So that the longer oscillations are damped out the quicker, a lucky fact indeed for aeronautic.

The study of these free oscillations of an aeroplane by Lanchester, Bryan and a host of others has given sufficient data for aeronautic engineers to build stable aeroplane but one point of danger still remains which has been the doom of many an aviator. It is well known that if a periodic disturbance acts on a system which has its own natural period of oscillation and if the periods of the disturbance be almost equal or equal to the natural period, the system may be thrown in a violent state of oscillation which may prove dangerous. It is one redeeming feature in the motion of an aeroplane that its oscillations are damped which may sometimes check the abovementioned pernicious tendency. In order to study this problem mathematically I have undertaken the following work. It has been found that under certain qualifying conditions the aeroplane may have a stable motion in the face of such a periodic gust of wind. The problem could not be very thoroughly treated as experimental datas and theoretical knowledge of air forces is still meagre in spite of the enormous strides the science has taken during and since the war in the hands of Prandtl, Eiffel and Bairstow.

We start by writing down the general equations of motions of Rigid Dynamics. Taking the centre of mass of the aeroplane as the origin of co-ordinates and 3 rectangular axes fixed relatively to the aeroplane and moving with it in space and using the following notations

W,	weight of the aeroplane.
A, B, C,	moments of inertia about the axes.
D, E, F,	corresponding products of inertia.
u, v, w	components of translational velocity.
p, q, r	„ of angular velocity.
h_1. h_2, h_3	„ of angular momentum.

we have the following equations of motion :

$$W\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)=\text{Acc. force along the } x\text{–axis}$$

and two similar equations, also

$$\frac{dh_1}{gdt} + \frac{qh_3}{q} - \frac{rh_2}{g} = \text{Acc. torque about the } x\text{—axis}$$

and two similar equations, and

$$h_1 = \text{A}p - \text{F}q - \text{E}r$$

$$h_2 = \text{B}q - \text{D}r - \text{F}p$$

$$h_3 = \text{C}r - \text{E}p - \text{D}q$$

In the first place, let the aeroplane be flying steadily in a horizontal straight line. Let this be the axis of x (the line parallel to the line of flight and passing through the C. G.) and a line drawn vertically downwards through the C-G, the y—axis and a horizontal line perpendicular to these the axis of x.

If the aeroplane be turned in any other directions the following angular co-ordinates will specify them :

Starting from an initial position, let us rotate the aeroplane about the y—axis through an angle ψ and then about the new position of the axis of z through an angle θ and lastly about the final position of the x—axis through an angle ϕ. The cosines of the angles between the old axis $x_0y_0z_0$ and the new $x_1y_1z_1$ an given by

	x_1	y_1	z_1
x_0	$\cos\theta\cos\psi$,	$\sin\phi\sin\psi - \cos\phi\cos\psi\sin\theta$,	$\cos\phi\sin\psi + \sin\phi\cos\psi\sin\theta$.
y_0	$\sin\theta$,	$\cos\theta\cos\phi$,	$-\cos\theta\sin\phi$.
z_0	$-\cos\theta\sin\psi$,	$\sin\phi\cos\psi + \cos\phi\sin\psi\sin\theta$,	$\cos\phi\cos\psi - \sin\phi\sin\psi\sin\theta$.

and the angular velocities p, q, r are given in terms of θ, ϕ, ψ

$$p = \dot{\phi} + \dot{\psi}\sin\theta$$

$$q = \dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi$$

$$r = \dot{\theta}\cos\phi - \dot{\psi}\cos\theta\sin\phi$$

The imposed forces and couples are due to (*i*) gravity (*ii*) the propeller thrust (*iii*) air resistances (*iv*) the periodic disturbances due to the gusts of wind.

The components of gravity along the axes are W $\sin\theta$, W $\cos\theta \cos\phi$, $-$W $\cos\theta \sin\phi$ and and the corresponding moments all vanishing.

The propeller thrust is assumed to act along a line parallel to the x—axis and at a point on the y—axis distant 'h' from the origin, then the components of thrusts are

Point of application,	O, h, O
Force,	H, O, O
Torque,	O, O, $-$Hh

For the components of air resistances we assume that they reduce to X Y Z and L M N and these are taken positive when they tend to retard the corresponding motions of translation and rotations. The components of periodic gusts are Pe^{mt}, Qe^{mt}, $R'e^{mt}$, $P'e^{mt}$, $Q'e^{mt}$, Re^{mt} where 'm' is an imaginary quantity. Hence the equations of motion are in the case of symmetrical aeroplane (in which D$=$E$=$0)

$$\frac{W}{g}\left(\frac{du}{dt}+qw-rv\right)=W\sin\theta+H-X-Pe^{mt}$$

$$\frac{W}{g}\left(\frac{dv}{dt}+ru-pw\right)=W\cos\theta\cos\phi-Y-Qe^{mt}$$

$$\frac{W}{g}\left(\frac{dw}{dt}+pv-qu\right)=-W\cos\theta\sin\phi-Z-R'e^{mt}$$

$$\frac{A}{g}\frac{dp}{dt}-\frac{F}{g}\cdot\frac{dq}{dt}+(C-B)\frac{rq}{g}+F\frac{pr}{g}=-L-P'e^{mt}$$

$$\frac{B}{g}\frac{dq}{dt}-\frac{F}{g}\frac{dp}{dt}+(A-C)\frac{pr}{g}-F\frac{qr}{g}=-M-Q'e^{mt}$$

$$\frac{C}{g}\frac{dr}{dt}+(B-A)\frac{pq}{g}-F\frac{p^2-q^2}{g}=-Hh-N-Re^{mt}$$

Now suppose that the aeroplane was descending with uniform velocity U in the direction of the x—axis and let the axis make a constant angle θ_0 with the horizon, before the periodic gust began to operate ; then initially $u=$U, v, w, p, q, r zero, and the components of the gusts absent and let the components of air resistances in this case be denoted by $X_0 Y_0 Z_0$, $L_0 M_0 N_0$. The equations of steady motion are

$$\begin{aligned} 0 &= W \sin \theta_0 + H - X_0 \\ 0 &= W \cos \theta_0 \quad - Y_0 \\ 0 &= \quad - Z_0 \\ 0 &= \quad - L_0 \\ 0 &= \quad - M_0 \\ 0 &= \quad - Hh - N_0 \end{aligned}$$

If now the periodic gust begins to operate we assume that the velocity components become $U+u$, v, w, p, q, r when u, v, w, p, q, r are all small. In the theory of small oscillations of dynamics we suppose that the squares and products of these velocities are negligible. The resistances X, Y, Z, L, M, N are functions of the velocity components $U+u$, v, w, p, q, r and the further assumption in dealing with small oscillations is that to a first approximation these resistances are expressible in the form

$$X = X_0 + uX_u + vX_v + wX_w + pX_p + qX_q + rX_r \qquad \text{(Bryan)}$$

This assumption is common in treatise on theoretical mechanics as a first approximation when small oscillations are concerned. In our case since we have assumed the aeroplane as symmetrical

$$X = X_0 + uX_u + vX_v + rX_r$$

X_w, X_p, X_q being zero from considerations of symmetry. In small oscillations moreover θ will differ from θ_0 by a small quantity ϵ and ϕ will be small, then

$$\sin \theta = \sin \theta_0 + \epsilon \cos \theta, \qquad \cos \theta = \cos \theta_0 - \epsilon \sin \theta_0$$

$$\sin \phi = \phi \qquad \cos \phi = 1$$

Hence the modified equations of motion become

$$\frac{W}{g} \frac{du}{dt} = W(\sin \theta_0 + \epsilon \cos \theta_0) + H - X_0 - uX_u - vX_v - rX_r - Pe^{mt}$$

$$\frac{W}{g} \left(\frac{dv}{dt} + rU \right) = W(\cos \theta_0 - \epsilon \sin \theta_0) - Y_0 - uY_u - vY_v - rY_r - Qe^{mt}$$

$$\frac{W}{g} \left(\frac{dw}{dt} - qU \right) = -W\phi \cos \theta_0 - Z_0 - wZ_w - pZ_p - qZ_q - R'e^{mt}$$

$$\frac{A}{g}\frac{dp}{dt}-\frac{F}{g}\frac{dq}{dt}=L_0-wL_w-pL_p-qL_q-P'e^{mt}$$

$$\frac{B}{g}\cdot\frac{dq}{dt}-\frac{F}{g}\cdot\frac{dp}{dt}=M_0-wM_w-pM_p-qM_q-Q'e^{mt}$$

$$\frac{C}{g}\cdot\frac{dv}{dt}\qquad\qquad=-Hh-N_0-uN_u-vN_v-rN_r-Re^{mt}$$

We substitute from the equations of equilibrium and rearrange the equations in two groups, the first group containing those involving u, v, r, and the second group involving p, q, w We thus get

$$\frac{W}{g}\cdot\frac{du}{dt}=W\epsilon\cos\theta_0-uX_u-vX_v-rX_r-Pe^{mt}$$

$$\frac{W}{g}\left(\frac{dv}{dt}+rU\right)=-W\epsilon\sin\theta_0-uY_u-vY_v-rY_r-Qe^{mt}$$

$$\frac{C}{g}\cdot\frac{dr}{dt}=-uN_u-vN_v-rN_r-Re^{mt}$$

and the second group

$$\frac{W}{g}\left(\frac{dw}{dt}-qU\right)=-W\phi\cos\theta_0-wZ_w-pZ_p-qZ_q-R'e^{mt}$$

$$\frac{A}{g}\frac{dp}{dt}-\frac{F}{g}\frac{dq}{dt}=-wL_w-pL_p-qL_q-P'e^{mt}$$

$$\frac{B}{g}\frac{dq}{dt}-\frac{F}{g}\frac{dp}{dt}=-wM_w-pM_p-qM_q-Q'e^{mt}$$

The complementary functions of the first group of equations will give the longitudinal or symmetrical oscillations of the aeroplane and the second group give the lateral or transverse oscillations.

Group I.

To solve this group of equations we assume u, v and ϵ each proportional to u_0e^{mt}. v_0e^{mt}, ϵ_0e^{mt} (as is the convention in the case of forced oscillations) so that $\frac{du}{dt}=mu_0e^{mt}$, $\frac{dv}{dt}=mv_0e^{mt}$, $\frac{d\epsilon}{dt}=m\epsilon_0e^{mt}$ and since $\theta=\theta_0+\epsilon$, $\frac{d\epsilon}{dt}=\frac{d\theta}{dt}=m\epsilon=r$.

Hence substituting these values in the first group and eliminating e^{mt} we get

$$\left(\frac{W}{g}m+Xu\right)u_0+Xv\cdot v_0+(mX_r-W\cos\theta)\,\epsilon+P=0$$

$$Y_u\cdot u_0+\left(\frac{W}{g}m+Yv\right)v_0+\left\{\left(W\frac{U}{g}+Y_r\right)m+W\sin\theta\right\}\epsilon_0+Q=0$$

$$N_u\cdot u_0+N_v\cdot v_0+\left(\frac{C}{g}m^2+N_r\cdot m\right)\epsilon_0+R=0$$

Solving these equations for u_0, v_0, ϵ_0 we get

$$\frac{u_0}{\begin{vmatrix} X_\iota, & mX_r-W\cos\theta, & P \\ \frac{W}{g}m+Y_\iota, & \left(W\frac{U}{g}+Y_r\right)m+W\sin\theta, & Q \\ N_v, & \frac{C}{g}m^2+N_r\cdot m. & R \end{vmatrix}}$$

$$=\frac{-v_0}{\begin{vmatrix} \frac{W}{g}m+X_u. & mX,-W\cos\theta, & P \\ Y_v, & \left(\frac{W}{g}U+Y_r\right)m+W\sin\theta, & Q \\ N_v, & \frac{C}{g}m^2+N_r\cdot m & R \end{vmatrix}}$$

$$=\frac{\epsilon_0}{\begin{vmatrix} \frac{W}{g}m+X_u, & X_v, & P \\ Y_u, & \frac{W}{g}m+Y_v, & Q \\ N_u, & N_v, & R \end{vmatrix}}$$

$$=\frac{-1}{\begin{vmatrix} \frac{W}{g}m+X_u, X_v, & X_r\cdot m-W\cos\theta \\ Y_u, \frac{W}{g}m+Y_v, & \left(W\frac{U}{g}+Y_r\right)m+W\sin\theta \\ N_u, & N_v, \frac{C}{g}m^2+N_r\cdot m \end{vmatrix}}=\frac{-1}{F(m}$$

where $F(m)=A_0m^4+B_0m^3+C_0m^2+D_0m+E_0$ and these values of A_0, B_0, C_0, D_0 and E_0 are given by Bryan

$$A_0=CW^2$$

$$B_0/g=CW(X_uX_v+Y_v)+W^2\cdot N_r$$

$$C_0/g^2=C(X_uY_v-X_vY_u)+W[(Y_vN_r-Y_rN_v)+(X_uN_r-X_rN_u)]$$
$$-\frac{W^2}{g}U\cdot N_v$$

$$D_0/g^3=X_u(Y_vN_r-Y_rN_v)+X_v(Y_rN_u-Y_uN_r)+(Y_uN_v-Y_vN_u)X_r$$
$$+W\frac{U}{g}(X_rN_u-X_uN_r)+\frac{W^2}{g}(N_u\cos\theta-N_r\sin\theta)$$

$$E_0/g^4=\frac{W}{g}[-\cos\theta(Y_vN_u-Y_uN_v)-\sin\theta(X_uN_v-X_vN_u)]$$

Hence

$$u_0=\frac{\begin{vmatrix} X_v, & mX_r-W\cos\theta, & P \\ \frac{W}{g}+Y_v, & \left(W\frac{U}{g}+Y_r\right)m+W\sin\theta, & Q \\ N_v, & \frac{C}{g}m^2+N_r\cdot m, & R \end{vmatrix}}{A_0m^4+B_0m^3+C_0m^2+D_0m+E_0}=\frac{\phi(m)}{F(m)}$$

similarly $v_0=\dfrac{\psi(m)}{F(m)}$ and $\epsilon_0=\dfrac{\chi(m)}{F(m)}$; thus the complete values are

$$u=a_1e^{m_1t}+a_2e^{m_2t}+a_3e^{m_3t}+a_4e^{m_4t}+u_0e^{mt}$$

$$v=b_1e^{m_1t}+b_2e^{m_2t}+b_3e^{m_3t}+b_4e^{m_4t}+v_0e^{mt}$$

$$\epsilon=c_1e^{m_1t}+c_2e^{m_2t}+c_3e^{m_3t}+c_4e^{m_4t}+\epsilon_0e^{mt}$$

where m_1 m_2 m_3 m_4 are the roots of $F(m)=0$

The motion of the aeroplane is almost exactly the same in the above case as it was when the gust did not act, only a periodic oscillation has been superadded on the other oscillations of the system, Hence its conditions of stability in the general case can be found from Bryan. But if $m=m_1$ or m_2 or m_3 or m_4 i.e. if the period of the gust becomes identically equal to one of the periods of natural oscillations of the system then $F(m)=0$ and u_0 v_0 and ϵ_0 would become infitely great unless $\phi(m)$, $\psi(m)$ and $\chi(m)$ become zero at the same time i.e. unless $\phi(m)$ and $F(m)$, $\psi(m)$ and $F(m)$, $\chi(m)$ and $F(m)$ have common roots.

Now $\qquad F(m)=A_0m^4+B_0m^3+C_0m^2+D_0m+E_0$

$$\phi(m)=\qquad Am^3+Bm^2+Cm+D \text{ (say)}$$

If $F(m)$ and $\phi(m)$ have a common factor to find a relation between the coefficients of the two equations we follow Sylvester's Dialytic method of elimination. Now suppose certain value of m make both $F(m)$ and $\phi(m)$ zero i.e.

$$A_0m^4+B_0m^3+C_0m^2+D_0m+E_0=0 \qquad \dots \quad (1)$$

$$Am^3+Bm^2+Cm+D=0 \qquad \dots \quad (2)$$

Let us consider the different powers of m as so many distinct unknowns. We have then 2 non-homogenous linear equations in the four unknowns m, m^2, m^3, m^4, Multiplying (1) by m and then by m^2 and (2) by m, m^2, m^3 in turn we have

$$A_0m^6+B_0m^5+C_0m^4+D_0m^3+E_0m^2=0$$

$$A_0m^5+B_0m^4+C_0m^3+D_0m^2+E_0m=0$$

$$A_0m^4+B_0m^3+C_0m^2+D_0m+E_0=0$$

$$Am^3+Bm^2+Cm+D=0$$

$$Am^4+Bm^3+Cm^2+Dm=0$$

$$Am^5+Bm^4+Cm^3+Dm^2=0$$

$$Am^6+Bm^5+Cm^4+Dm^3=0$$

a system of seven non-homagenous, linear equations in six unknowns.

If m satisfy (1) and (2) it will evidently satisfy all the above equations. These equations are therefore consistent. Hence eliminating m^6, m^5, m^4, m^3, m^2, m, we get as a necessary condition for (1) and (2) to have a common root

$$\begin{vmatrix} A_0 & B_0 & C_0 & D_0 & E_0 & 0 & 0 \\ 0 & A_0 & B_0 & C_0 & D_0 & E_0 & 0 \\ 0 & 0 & A_0 & B_0 & C_0 & D_0 & E_0 \\ 0 & 0 & 0 & A & B & C & D \\ 0 & 0 & A & B & C & D & 0 \\ 0 & A & B & C & D & 0 & 0 \\ A & B & C & D & 0 & 0 & 0 \end{vmatrix} = 0$$

In order to avoid algebraic complications at the outset, we first consider a single lifting plane propelled horizontally by a central thrust. Two surfaces S_1 S_2 of which the front surface S_1 supports the whole weight of the aeroplane being inclined to the line of flight at an angle α, while the rear surface S_2 acts as a tail or rudder or auxiliary plane, being placed in a neutral direction (so that $\alpha_2=0$). Distance between the centres of pressure of the two planes is l, the line of action of the propellor thus passes through the centre of gravity of the machine; the direction of the thrust being along the line of flight is horizontal. The values of the nine derivations $X_u \ldots N_r$ are given by Bryan in this case as follows :

$$X_u=2KS_1U\sin^2\alpha, \qquad X_v=KS_1U\sin\alpha\cos\alpha, \qquad X_r=0$$

$$Y_u=2KS_1U\sin\alpha\cos\alpha, \quad Y_v=KS_1U\cos^2\alpha+KS_2U, \quad Y_r=-KS_2Ul$$

$$N_u=0 \qquad , \quad N_v=-KS_2Ml \qquad , \quad N_r=KS_2Ul^2$$

where K is the coefficient of resistance of the plane.

Hence we obtain in the case when the machine is descending with velocity U at an angle θ with the horizontal

$$A_0=CW^2$$

$$B_0/gU=CWK[S_1(1+\sin^2\alpha)+S_2]+W^2KS_2l^2$$

$$C_0/g^2U^2=2CK^2S_1S_2\sin^2\alpha+WK^2S_1S_2l^2(1+\sin^2\alpha)+\frac{W^2}{g}\,KS_2l^2$$

$$D_0/g^3U^3 = \frac{W}{g}\left(K^2S_1S_2l \sin\alpha\cos\alpha\cdot\overline{2\tan\alpha+\tan\theta}\right)$$

$$E_0/g^4U^4 = \frac{2W}{gU^2}K^2S_1S_2l\sin\alpha\cos(\alpha-\theta)$$

and for u_0

$$A = \frac{CW}{g^2}P$$

$$B = [CKS_1U\cos^2\alpha + CKS_2U + WKS_2Ul^2)\frac{P}{g} - CKS_1U\sin\alpha\cos\alpha\cdot\frac{Q}{g}$$

$$C = \left[K^2S_1S_2U^2l^2\cos^2\alpha + \frac{W}{g}KS_2U^2l\right]P - K^2S_1S_2U^2l^2\sin\alpha$$

$$\cos\alpha\cdot Q$$

$$+\left[KS_1U\sin\alpha\cos\alpha\left(W\frac{U}{g} - KS_2Ul\right) + \frac{W^2}{g}\cos\theta\right]R$$

$$D = KS_2Ul\cdot W\sin\theta\cdot P + KS_2Ul\ W\cos\theta\cdot Q$$

$$+[KS_1U\sin\alpha\cos\alpha\ W\sin\theta + W\cdot\cos\theta\ (KS_1U\cos^2\alpha + KS_2U)]R$$

and for the equation of equilibrium $W\cos\theta = KS_1U^2\sin\alpha\cos\alpha$

The quantities P, Q, R that we have assumed for the resolved parts of the magnitude of the gust will have a relation among themselves like $aP = bQ = cR$ where a, b, c in the general case are functions of time.

Case I.

In this case we shall assume that $a = b = c = 1$, $\theta = o$ i.e. the machine is flying horizontally and α a small quantity so that $\sin^2\alpha$ and higher powers of $\sin\alpha$ are neglected. Then

$$A_0 = CW^2$$

$$B_0/gU = CWK(S_1+S_2) + W^2KS_2l^2$$

$$C_0/U^2g^2 = WKS_1S_2l^2 + \frac{W^2}{g}KS_2l$$

$$D_0/U^3g^3 = 0$$

$$E_0/g^4U^4 = \frac{2W}{U^2g}K^2S_1S_2l\sin\alpha\cos\alpha$$

and for the equation of equilibrium $W=KS_1U^2 \sin\alpha\cos\alpha$

and

$$A=\frac{CW}{g^2}P$$

$$B=\frac{PKU}{g}\left[CS_1(1-\cos\alpha\sin\alpha)+S_2(C+Wl^2)\right]$$

$$C=P\Big[K^2S_1S_2U^2l^2(1-\cos\alpha\sin\alpha)+K^2S_1S_2^2U^2l\sin\alpha\cos\alpha\left(\frac{U^2}{g}-1\right)+2\frac{W^2}{g}\Big]$$

$$D=PWKU(S_2l+S_1+S_2)$$

Then expanding the Sylvester's Determinant and remembering that $W=KS_1U^2 \sin\alpha\cos\alpha$ for the condition of equilibrium and neglecting $\sin^2\alpha$ and higher powers of $\sin\alpha$, we find that the determinant reduces to zero i.e. $\phi(m)$ and $F(m)$ have a common factor between them.

Now $u_0=\dfrac{\phi(m)}{F(m)}$ and even if $m=m_1$ i.e. if a period of the gust coincide with a period of the natural oscillation of the system, u_0 will not tend to become infinitely great i.e. the forced oscillation would not make the system unstable for that particular period of the gust.

We have arrived at this particular result by assuming α small i.e. by neglecting $\sin^2\alpha$ and higher powers of $\sin\alpha$. Bryan has shown that the natural oscillation of the system that result from the above assumption gives the short oscillations of the aeroplane. Hence if the period of the gust of wind coincide with the period of the small oscillations of the aeroplane it will have no effect in violently disturbing the stability of the system so far as the velocity u is concerned. Hence $u=a_1e^{m_1t}+a_2e^{m_2t}+a_3e^{m_3t}+a_4e^{m_4t}+u_0e^{mt}$ gives the natural and force oscillations of the system in the direction of its motion so far as the short oscillation are concerned. The case when the period of the gust coincides with the periods of the long or slow oscillations we can no longer neglect $\sin^2\alpha$ only and we shall take it up afterwards first finishing the examination of the effect on v_0 and ϵ_0. The case of v_0 is quite similar to that of u_0 but ϵ_0 both because it is an angle whose great variation may very well become dangerous to the aeroplane and it presents quite a different type of equations for $\chi(m)$, will give some interesting results. We have

$$\epsilon_0 = \frac{\chi(m)}{F(m)} \text{ where}$$

$$\chi(m) = P \begin{vmatrix} \frac{W}{g} m + X_u, & X_r, & 1 \\ Y_u, & \frac{W}{g} m + Y_r, & 1 \\ N_u, & N_r. & 1 \end{vmatrix} = Am^2 + Bm + C$$

$$A = P \frac{W^2}{g^2}, \ B = -P \frac{W}{g} (N_u + N_r), \ C = P[(Y_u N_r - N_u Y_r) \\ -(X_u N_v - X_r N_u) - X_0 Y_u]$$

Hence

$$A_0 = CW^2$$

$$B_0/gU = CWK(S_1 + S_2) + W^2 KS_2 l^2$$

$$C_0/g^2 U^2 = WKS_1 S_2 l + \frac{W^2}{g} KS_2 l$$

$$D_0 = 0 \qquad , \qquad E_0/g^4 U^4 = \frac{2W}{gU^2} K^2 S_1 S_2 l \sin a \cos a$$

and

$$A = P \frac{W^2}{g^2}, \ B = P \frac{W}{g} KS_2 Ul, \ C = -2PK^2 S_1 S_2 U^2 l \sin a \cos a$$

If a period of the gust of wind be equal to the natural period of oscillation of the aeroplane then $m = m_1$ and to find the condition that $F(m)$ and $\chi'm)$ may have a common factor. The equations are

$$A_0 m^4 + B_0 m^3 + C_0 m^2 + D_0 m + E_0 = 0$$

$$Am^2 + Bm + C = 0$$

The condition these two equations have a common root is given by Bôacher as follows:

The resultant R of the two equations must be zero i.e.

$$R=\begin{vmatrix} A_0 & B_0 & C_0 & D_0 & E_0 & 0 \\ 0 & A_0 & B_0 & C_0 & D_0 & E_0 \\ 0 & 0 & 0 & A & B & C \\ 0 & 0 & A & B & C & 0 \\ 0 & A & B & C & 0 & 0 \\ A & B & C & 0 & 0 & 0 \end{vmatrix}=0$$

Expanding R and remembering that $W=KS_1U^2 \sin a \cos a$ and neglecting terms containing $\sin^2 a$ and higher powers of $\sin a$ we see that $R=0$ i.e., $F(m)$ and $\chi(m)$ have a common factor between them. Thus

$$\epsilon=c_1e^{m_1t}+c_2e^{m_2t}+c_3e^{m_3t}+c_4{}^{m_4t}+\epsilon_0e^{mt}$$

represents oscillatory motions in all cases except when m coincides with those values of the roots of $F(m)=0$ that give slow oscillations of the system. Now since $F(m)$ and $\chi(m)$ have a common factor corresponding to that value of $m=m_1$ which gives the short oscillations of the system, the degree of that common factor is two, hence it is proportional to $\chi(m)$ i.e. $F(m)=\chi(m)F_1(m)$ and $\epsilon_0=\frac{a}{F_1(m)}$; those two values of m that make $F_1(m)=0$ will make ϵ_0 infinitely great i.e., make the motion of the aeroplane unstable. Hence those values of m corresponding to the long oscillations of the system will give the aeroplane a pitching tendency which may prove dangerous.

In the case of u_0 and v_0 when $\sin^2 a$ does not vanish we see that the Sylvester's Determinant does not vanish i.e. $F(m)$ and $\phi(m)$, $\psi(m)$ have not a comon root. Therefore when the period of the gust coincides with that of the slow oscillations of the system u_0 and v_0 become very great but this fact by itself could not have given the machine any instability. It is only when we have considered the value of ϵ_0 in this case and find as above that it also is very great we conclude that the forced oscillations in this case assume dangerous proportion.

Hence in the case when the machine is flying horizontally the effect of a periodic gust of wind will be only to superimpose another oscillation on the system (provided the machine has inherent stability) for all periods of its gust except when it synchronises with the long oscillations of the aeroplane.

CASE II.

In this case we still assume $a=b=c=1$ but $\theta \neq 0$ i.e. the machine is descending at an angle θ to the horizontal and for purposes of approximation and algebraic simplifications we shall suppose θ and a both small so that $\cos(\theta-a)$ may be taken equal to unity, then

$$A_0 = CW^2.$$

$$B_0/gU = CWK[S_1(1+\sin^2 a)+S_2]+W^2KS_2l^2$$

$$C_0/g^2U^2 = 2CK^2S_1S_2 \sin^2 a + WK^2S_1S_2l^2(1+\sin^2 a)+\frac{W^2}{g}\ KS_2l$$

and for D_0 and E_0 we must go to the equations for u_0, v_0 and ϵ_0 in which substituting the values of X_u $X_v \dots N_r$, we get

$$(W\frac{m}{g}+2KS_1U \sin^2 a)u_0+KS_1U \sin a \cdot v_0 - W \cos\theta \cdot \epsilon_0 + P = 0$$

$$2KS_1U \sin a \cos a \cdot u_0 + \left(W\frac{m}{g}+KS_1U \cos^2 a + KS_2U\right)v_0$$

$$+\left[\left(W\frac{U}{g}-KS_2Ul\right)m + W \sin\theta\right]\epsilon_0 + P = 0$$

$$O - KS_2Ul \cdot v_0 + \left(\frac{C}{g}\ m^2 + KS_2Ul^2mv\right)\epsilon_0 \qquad + P = 0$$

Hence

$$D_0/g^3U^3 = \frac{2W}{g}\ K^2S_1S_2l \sin^2 a + \frac{W^2}{gU^2}KS_2l \sin\theta$$

$$E_0/g^4U^4 = \frac{2W}{gU^2}K^2S_1S_2l[\sin a \cos a \cos\theta + \sin^2 a \sin\theta]$$

From the equations of equilibrium W cos $\theta = KS_1U^2 \sin\alpha \cos\alpha$ these reduce to

$$D_0/g^3U^3 = \frac{W}{g} K^2S_1S_2 l \sin\alpha \cos\alpha\ (2 \tan\alpha + \tan\theta)$$

$$E_0/g^4U^4 = \frac{2W}{U^2g} K^2S_1S_2 l \sin\alpha \cos(\theta - \alpha)$$

If now we neglect $\sin^2\alpha$ and higher powers of $\sin\alpha$

$$A_0 = CW^2$$

$$B_0/gU = CWK(S_1 + S_2) + W^2KS_2l^2$$

$$C_0/g^2U^2 = WK^2S_1S_2l^2 + \frac{W^2}{g} KS_2l$$

$$D_0/g^3U^3 = \frac{W}{g} K^2S_1S_2l \sin\alpha \cos\alpha \cdot \tan\theta$$

$$E_0/g^4U^4 = \frac{2W}{U^2g} K^2S_1S_2l \sin\alpha$$

We see that A_0 B_0 C_0 and E_0 are the same in this case as in Case I and in $\chi(m)$, θ does not enter, hence ϵ_0 finite for those values of m that do not coincide with a root of $F(m)=0$ giving the slow oscillation of the system.

But in the case of κ_0

$$A = C \frac{W}{g^2} \cdot P$$

$$B = \frac{PKU}{g} [CS_1(1 - \cos\alpha \sin\alpha) + S_2(C + Wl^2)]$$

$$C = P\Big[K^2S_1S_2U^2l^2(1 - \sin\alpha \cos\alpha) + \frac{W}{g} KU^2(S_2l - S_1 \sin\alpha \cos\alpha)$$

$$- K^2S_1S_2U^2l \sin\alpha \cos\alpha + \frac{W}{g} KS_1U^2 \sin\alpha \cos\alpha \Big]$$

$$=\mathrm{P}\Big[\mathrm{K}^2\mathrm{S}_1\mathrm{S}_2\mathrm{U}^2l^2(1-\sin\alpha\cos\alpha)-\mathrm{K}^2\mathrm{S}_1\mathrm{S}_2\mathrm{U}^2l\sin\alpha\cos\alpha+\frac{\mathrm{W}}{g}\mathrm{KS}_2\mathrm{U}^2l\Big]$$

$$\mathrm{D}=\mathrm{PKU}[(\mathrm{S}_2l+\mathrm{S}_1\sin\alpha\cos\alpha)\mathrm{W}\sin\theta+(\mathrm{S}_2l+\mathrm{S}_1+\mathrm{S}_2)\mathrm{W}\cos\theta]$$

and the equation of equilibrium $\mathrm{W}\cos\theta=\mathrm{KS}_1\mathrm{U}^2\sin\alpha\cos\alpha$. We find on expanding R that it does not vanish even when $\sin^2\alpha$ and higher powers of $\sin\alpha$ are neglected. Hence in this case the machine is distinctly unstable for the forced oscillations corresponding to either periods of natural oscillations.

Group II.

At the outset in this group we start with the assumption that $\mathrm{P}'=\mathrm{Q}'=\mathrm{R}'=$ a constant quantity which is conveniently taken as unity.

To solve this group of equations we assume w, p, q, each proportional to w_0e^{mt}, p_0e^{mt}, q_0e^{mt}, so that $\frac{dw}{dt}=w_0me^{mt}$, $\frac{dp}{dt}=p_0me^{mt}$, $\frac{dq}{dt}=q_0me^{mt}$. Hence substituting these in this group and eliminating e^{mt} we get

$$\Big(\frac{\mathrm{W}}{g}m+\mathrm{Z}_w\Big)w_0+\Big(\frac{\mathrm{W}}{m}\cos\theta+\mathrm{Z}_p\Big)p_0$$
$$+\Big(-\mathrm{W}\frac{\mathrm{U}}{g}-\frac{\mathrm{W}}{m}\sin\theta+\mathrm{Z}_q\Big)q_0+\mathrm{R}'=0$$

$$\mathrm{L}_w\cdot w_0+\Big(\frac{\mathrm{A}}{g}m+\mathrm{L}_p\Big)p_0+\Big(-\mathrm{F}\frac{m}{g}+\mathrm{L}_q\Big)q_0+\mathrm{P}'=0$$

$$\mathrm{M}_w\cdot w_0+\Big(-\mathrm{F}\frac{m}{g}+\mathrm{M}_p\Big)p_0+\Big(\mathrm{B}\frac{m}{g}+\mathrm{M}_q\Big)q_0+\mathrm{Q}'=0$$

on the following identity $m\phi\cos\theta=p\cos\theta-q\sin\theta$.

Solving these equations for w_0. p_0, q_0 and putting $P'=Q'=R'=1$

$$\frac{w_0}{\begin{vmatrix} \frac{W}{m}\cos\theta+Z_p, & -W\frac{U}{g}-\frac{W}{m}\sin\theta+Z_p, & 1 \\ A\frac{m}{g}+L_p, & -F\frac{m}{g}+L_q, & 1 \\ -F\frac{m}{g}+M_p, & B\frac{m}{g}+M_q, & 1 \end{vmatrix}} = \frac{-p_0}{\begin{vmatrix} W\frac{m}{g}+Z_w, & -W\frac{U}{g}-\frac{W}{m}\sin\theta+Z_q, & 1 \\ L_w, & -F\frac{m}{g}+L_q, & 1 \\ M_w, & B\frac{m}{g}+M_q, & 1 \end{vmatrix}}$$

$$= \frac{q_0}{\begin{vmatrix} W\frac{m}{g}+Z_w, & \frac{W}{m}\cos\theta+Z_p, & 1 \\ L_w, & A\frac{m}{g}+L_p, & 1 \\ M_w, & -F\frac{m}{g}+M_p, & 1 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} W\frac{m}{g}+Z_w, & \frac{W}{m}\cos\theta+Z_p, & -W\frac{U}{g}-\frac{W}{m}\sin\theta+Z_q \\ L_w, & A\frac{m}{g}+L_p, & -F\frac{m}{g}+L_q \\ M_w, & -F\frac{m}{g}+M_p, & B\frac{m}{g}+M_q \end{vmatrix}}$$

Hence

$$w_0=\frac{Am^3+Bm^2+Cm+D}{A_1m^4+B_1m^3+C_1m^2+D_1m+E_1}\,.$$

where

$$A_1=W(AB-F^2)$$

$$B_1/g=Z_q(AB-F^2)+W[AM_q+BL_p+F(L_q+M_p)]$$

$$C_1/g^2=Z_w[AM_q+BL_p+F(L_q+M_p)]+W(L_pM_q-L_qM_p)$$

$$-Z_p(FM_w+BL_w)-\left(Z_q-W\frac{U}{g}\right)(FL_w+AM_w)$$

$$D_1/g^3=Z_w(L_pM_q-L_qM_p)+Z_p(L_qM_w-M_qL_w)$$

$$+\left(Z_q-W\frac{U}{g}\right)(L_wM_p-L_pM_w)$$

$$+\frac{W}{g}[(FL_w+AM_w)\sin\theta-(BL_w+FM_w)\cos\theta]$$

$$E_1/g^4=\frac{W}{g}[(L_qM_w-M_qL_w)\cos\theta-(L_wM_p-M_wL_p)\sin\theta]$$

and

$$A=\frac{AB-F^2}{g^2}$$

$$B=\frac{1}{g}\Big[A(M_q-Z_q)+B(L_p-Z_q)+F(L_q-Z_q)+F(M_p-Z_p)$$

$$+W\frac{U}{g}(A+F)\Big]$$

$$C=(L_pM_q-L_qM_p)+(Z_pL_q-Z_qL_p)+(M_pZ_q-M_qZ_p)$$

$$+W\frac{U}{g}(L_p-M_p)+\frac{W}{g}[(A+F)\sin\theta-(B+F)\cos\theta]$$

$$D=W\cos\theta\,(L_q-M_q)+W\sin\theta\,(L_p-M_p).$$

Also

$$p_0=\frac{A'm^3+B'm^2+C'm+D'}{A_1m^4+B_1m^3+C_1m^2+D_1m+E^1}$$

where

$$A' = (B+F)\,\frac{W}{g^2}$$

$$B' = \frac{1}{g}[B(Z_w - L_w) + F(Z_w - M_w) + W(M_q - L_q)]$$

$$C' = (L_q M_w - L_w M_q) + (M_q Z_w - M_w Z_q) + (L_w Z_q - Z_w L_q)$$

$$+ W\frac{U}{g}(M_w - L_w)$$

$$D' = (M_w - L_w)\,W \sin\theta.$$

Also

$$A'' = (A+F)\frac{W}{g^2}$$

$$B'' = \frac{1}{g}\,[A(Z_w - M_w) + F(Z_w - L_w) + W(L_p - M_p)]$$

$$C'' = (L_w M_p - L_p M_w) + (M_w Z_p - Z_w M_p) + (Z_w L_p - L_w Z_p)$$

$$D'' = (M_w - L_w)\,W \cos\theta$$

and

$$q_0 = \frac{A''m^3 + B''m^2 + C''m + D''}{A_1 m^4 + B_1 m^3 + C_1 m^2 + D_1 m + E_1}$$

CASE I.—STRAIGHT PLANES.

By this we mean planes perpendicular to the plane of $x-y$ with no vertical fins. In this case we get from Bryan Art. 77 Z_w, Z_p, Z_q, L_w, M_w. each=zero. Thus we find that $D_1=0$, $E_1=0$ and for straight planes $C_1=0$ (but in this case the original motion is unstable so we neglect it). If there are 2 planes whose angles of attack are α_1 and α_2 and the moments, of inertia about the plane $x-y$ be I_1 and I_2 we find

$$\frac{C_1}{K^2U^2g^2} = 2WI_1I_2\sin^2(\alpha_1 - \alpha_2)$$

But even in this case 2 roots of the biquadratic are zero which indicates lack of inherent stability, so we neglect this case altogether.

CASE II.

Now at once let us go to the system that is most stable and whose range of stability is great. This is the system with 2 raised fins at the same height. In this case we take 2 fins T_1 and T_2 (of total area T) one in front and other in the rear of the C. G. of the system and both above the x—axis in the x—y plane with the y of the C. P. equal and their joint C. P. in a line through the C. G. of the system perpendicular to the main planes. In this case, Bryan has shown, Art. 84, that the machine has inherent stability. Now to find the effect of the periodic gust on such a system.

Let (x, y) be the co-ordinates of the centre of mean position (or centre of pressure) of the 2 fins, and M_1, M_2, P the moments and the products of inertia of the areas of the fins with respect to axes parallel to the co-ordinate axes through (x, y) we get from Bryan since $M_1 = P = 0$, in this case for the fins

$$Z_w = K^1TU, \qquad Z_p = K^1TUy, \qquad Z_q = -K^1TUx$$

$$L_w = K^1TUy, \qquad L_p = K^1TUy^2, \qquad L_q = -K^1TUxy$$

$$M_w = -K^1TUx, \qquad M_p = -K^1TUxy, \qquad M_q = K^1U(Tx^2 + M_2)$$

and by Lanchester's 'Fin Resolution' $M_2 = \dfrac{T_1T_2}{T_1+T_2} \times$ (distance between fins)^2.

For simplification of algebra let us assume that $K^1 = K$ i.e. the coefficiants of resistance of the fins and the main planes are equal.

Also let us assume α small, so that $x = 0$ and also $F = 0$ i.e. the x-axis is a principal axis. Then

$$Z_w = KTU, \qquad Z_p = KUTy, \qquad Z_q = 0$$

$$L_w = KTUy, \qquad L_p = KUTy^2, \qquad L_q = 0$$

$$M_w = 0, \qquad M_q = 0, \qquad M_q = KUM_2$$

Also let us first consider the case when $\theta = 0$ i.e. the machine is flying horizontally.

The above are the resistance derivatives due to the 2 fins only, the resistance derivatives for a main plane at an angle α and a rudder plane are

$$Z_w=0, \quad Z_p=0, \quad Z_q=0$$

$$L_w=0, \quad L_p=\mathrm{KUI}\cos^2\alpha, \quad L_q=-2\mathrm{KUI}\sin\alpha\cos\alpha$$

$$M_w=0, \quad M_p=-\mathrm{KUI}\sin\alpha\cos\alpha, \quad M_q=2\mathrm{KUI}\sin^2\alpha$$

so that the whole resistance derivatives are, neglecting $\sin^2\alpha$ and higher powers of $\sin\alpha$

$$Z_w=\mathrm{KTU}, \quad Z_p=\mathrm{KUT}y, \quad Z_q=0$$

$$L_w=\mathrm{KUT}y, \quad L_p=\mathrm{KUT}y^2, \quad L_q=2\mathrm{KUI}\tan\alpha$$

$$M_w=0, \quad M_p=-\mathrm{KUI}\sin\alpha\cos\alpha. \quad M_q=\mathrm{KUM}_2$$

and

$$A_1=\mathrm{ABW}$$

$$B_1/g\mathrm{KU}=\mathrm{W}(\mathrm{AM}_2+\mathrm{BT}y^2)$$

$$C_1/g^2\mathrm{K}^2\mathrm{U}^2=\mathrm{TM}_2(\mathrm{A}+\mathrm{W}y^2)$$

$$D_1/g^3\mathrm{K}^3\mathrm{U}^3=\frac{\mathrm{W}}{g}\left[\mathrm{I}\cdot y\cdot\mathrm{T}\cdot\tan\alpha-\mathrm{B}\cdot\mathrm{T}\cdot\mathrm{Y}/\mathrm{K}^2\mathrm{U}^2\right]$$

$$E_1/g^4\mathrm{K}^4\mathrm{U}^4=-\frac{\mathrm{W}}{g}\cdot\mathrm{Y}\cdot\mathrm{T}\cdot\mathrm{M}_2/\mathrm{K}^2\mathrm{U}^2$$

and for p_0

$$A'=\frac{\mathrm{BW}}{g^2}$$

$$B'/\mathrm{KU}=\frac{1}{g}\left[\mathrm{BT}(1-y)+\mathrm{W}(\mathrm{M}_2+2\mathrm{I}\tan\alpha)\right]$$

$$C'/\mathrm{K}^2\mathrm{U}^2=\mathrm{TM}_2(1-y)+2\,\mathrm{I}\,\mathrm{T}\tan\alpha-\frac{\mathrm{W}}{kg}\mathrm{T}\cdot y\cdot$$

$$D'=0\ .$$

Using these quantities in the Sylvester's Determinant and remembering that $\mathrm{W}=\mathrm{KS}_1\mathrm{U}^2\sin\alpha\cos\alpha$ we find on developing and neglecting terms containing $\sin^2\alpha$ and higher powers of $\sin\alpha$, that the determinant vanishes i.e. the two equations

$$A_1m^4+B_1m^3+C_1m^2+D_1m+E_1=0$$

$$A'm^3+B'm^2+C'm+D'=0$$

have a common root between themselves. Now since $D'=0$ the last equation reduces to $A'm^2+B'm+C'=0$ and it will have two imaginary roots if y be negative, which is one of the conditions of the inherent stability of the machine. Also we know that there is only one type of lateral oscillation of the system i.e. $A_1m^4+B_1m^3+C^1m^2+D_1m+E_1=0$ has only a pair of imaginary roots and since a period of the gust is to coincide with a period of the oscillation of the aeroplane it must coincide with this particular root. Thus we see that the forced lateral oscillation is never unstable in the case of p_0 when the machine is flying horizontally and we can neglect $\sin^2 a$ and higher powers of $\sin a$. The same conclusion holds good in the case of w_0 and q_0.

Hence we conclude that when the plane is flying horizontally and a is such that $\sin^2 a$ and higher powers of $\sin a$ are negligible the effect of a periodic gust of wind on lateral motion is only to superimpose another oscillation which never becomes dangerous.

Case III.

In this we assume $\theta\neq 0$ but we neglect $\sin^2 a$ and higher powers of $\sin a$ then

$$A_1=ABW$$

$$B_1/KUg=W(AM_2+BTy^2)$$

$$C_1/K^2U^2g^2=TM_2(A+Wy^2)$$

$$D_1/K^3U^3g^3=\frac{W}{g}\,[y\mathrm{l}T\tan a-BTy\cos\theta/K^2U^2]$$

$$F_1/K^4U^4g^4=-\frac{W}{g}\,\frac{y}{K^2U^2}-TM_2\cos\theta$$

Thus we see that θ comes only in the form of cosine and that only in D_1 and E_1.

If we consider the case of p_0, A', B', C' remain the same as before only $D'=-KUTy\cdot W\sin\theta$; so also in q_0 only change is in D'' which becomes $D''=-KUTyW\cos\theta$ and in w_0, A and B remain the same C and D change so that

$$C=K^2U^2M_2Ty(y-1)-2K^2U^2Ty\mathrm{l}\tan a+\frac{W}{g}\,KU^2Ty^2$$

$$+\frac{W}{g}\,(A\sin\theta-B\cos\theta)$$

$$D=KU(Ty^2\sin\theta-M_2\cos\theta)\,W,$$

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1

ON A FACTORABLE CONTINUANT

BY

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Some factorable continuants have been discovered by such eminent mathematicians as Painvin,[1] Sylvester,[2] Metzler,[3] Muir[4] and Datta.[5] The present paper contains a continuant of the same class which has been derived from a finite series with the help of Heilermann's Theorem.[6] This continuant has been evaluated determinantally and some algebraic relations *viz.*, theorems (1), (2), (8), (9), (11), (12), (13), (14), (15) and (24) have been deduced. In converting the finite series to a continued fraction, we have come to a kind of determinants whose numerators and denominators are both resolvable into a number of binomial factors.[7]

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[1] Painvin, (1858). "Sur uncertain systeme d'quations lineaires." *Journ. de Math.* (2) III, pp. 41-46; or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, pp. 422-434.

[2] Sylvester J. J. (1854) "Theoreme sur les determinants de M. Sylvester." *Nouv Annales de Math.*, xiii, p. 305, or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, p. 425.

[3] W. H. Metzler. "Some factorable continuants," *Edin. Proc. Roy. Soc.*, 34, 1914 (223-229).

[4] Muir T. "Continuants resolvable into linear factors" *Trans. Edin. Roy. Soc.*, 41, 1905 (343-358). Muir T. "Factorizable continuants" *Trans. S. Afric. Philos, Soc.*, 15 pt. 1, 1904 (29-33).

[5] Haripada Datta, "On the Failure of Heilermann's Theorem." *Proc. Edin. Math. Soc.* Vol. 35. "On the Theory of Continued Fractions" *Proc. Edin. Math. Soc.*, Vol. 34.

[6] *Journal fur Math.* 33 (1845), p. 174.

[7] Cauchy, (1841) "Mémoire sur les fonctions alternées et sur les sommes alternées," Exercices d'analyse et de Phys. Math., II, pp. 151-159, or The Theory of determinants in the Historical Order of Developments by Muir T, Vol. I, pp. 342-345.

1.[1]

$$\{(1+y)(1+ay)(1+a^2y)\ldots(1+a^{r-1}y)\}$$

$$\equiv\{(y-\delta)(ay-\delta)(a^2y-\delta)\ldots(a^{r-1}y-\delta)\}$$

$$+{}^{r}S_1(1+\delta)\{(y-\delta)(ay-\delta)\ldots(a^{r-2}y-\delta)\}$$

$$+{}^{r}S_2\left(\frac{1}{a}+\delta\right)(1+\delta)\{y-\delta)(ay-\delta)\ldots(a^{r-3}y-\delta)\}+\ldots$$

$$+{}^{r}S_r\left\{\left(\frac{1}{a^{r-1}}+\delta\right)\left(\frac{1}{a^{r-2}}+\delta\right)\ldots(1+\delta)\right\} \qquad \ldots \quad (1)$$

where ${}^{r}S_p$ denotes the sum of the products of r factors $1, a, a^2, \ldots a^{r-1}$ taken p of them at a time.

Proof. Let us take the series.

$$(a+\delta)(a^2+\delta)(a^3+\delta), \qquad (1+\delta)(a+\delta)(a^2+\delta),$$

$$\left(\frac{1}{a}+\delta\right)(1+\delta)(a+\delta), \quad \left(\frac{1}{a^2}+\delta\right)\left(\frac{1}{a}+\delta\right)(1+\delta)\ldots$$

and obtain from it Δ_1, Δ_2 etc., the successive orders of differences by using $1, a, a^2\cdots$ as multipliers (see Art 5, Paper 3).

Then we shall find that in this particular case where there are three factors in each term of the original series, each term of Δ_4 and higher orders of differences vanishes. So by Art 5 (i), Paper 3 we have

$$(a+\delta)(a^2+\delta)(a^3+\delta)-{}^{k}S_1(1+\delta)(a+\delta)(a^2+\delta)$$

$$+{}^{k}S_2\left(\frac{1}{a}+\delta\right)(1+\delta)(a+\delta)-\ldots$$

$$+(-1)^k\,{}^{k}S_k\left(\frac{1}{a^{k-1}}+\delta\right)\left(\frac{1}{a^{k-2}}+\delta\right)\left(\frac{1}{a^{k-3}}+\delta\right)=0, \text{ where } k= \text{ or} >4.$$

Thus generally we have

$$\{(a+\delta)(a^2+\delta)\ldots(a^{r-1}+\delta)\}-{}^{k}S_1\{(1+\delta)(a+\delta)\ldots(a^{r-2}+\delta)\}$$

$$+{}^{k}S_2\left\{\left(\frac{1}{a}+\delta\right)(1+\delta)\ldots(a^{r-3}+\delta)\right\}-\ldots$$

$$+(-1)^k\,{}^{k}S_k\left\{\left(\frac{1}{a^{k-1}}+\delta\right)\left(\frac{1}{a^{k-2}}+\delta\right)\ldots(a^{r-k-1}+\delta)\right\}=0 \quad \ldots \quad (2)$$

where k=or $>r$.

---

[1] *cf.* "On the Evaluation of Some Factorable Continuants," Part II, Art 2. *Bul. Cal. Math. Soc.*, Vol. XIV, pp. 91-106. In subsequent references, this paper will be called Paper 3.

If the original series be 1, 1, 1, ..., then each term of Δ_1, Δ_2... is zero. Hence we have

$$1-{}^kS_1+{}^kS_2-\ldots+(-1)^k\,{}^kS_k=0$$

where k=or >1.

Let us now take the particular case of the theorem (1) when $r=3$ *viz.*,

$$\begin{aligned}(1+y)(1+ay)(1+a^2y) &\equiv (y-\delta)(ay-\delta)(a^2y-\delta)\\ &+{}^3S_1(1+\delta)(y-\delta)(ay-\delta)+{}^3S_2\left(\frac{1}{a}+\delta\right)(1+\delta)(y-\delta)\\ &+{}^3S_3\left(\frac{1}{a^2}+\delta\right)\left(\frac{1}{a}+\delta\right)(1+\delta).\qquad\ldots\quad(4)\end{aligned}$$

If we substitute δ, -1, $-\frac{1}{a}$ or $-\frac{1}{a^2}$ for y in (4), we can show by (2) and (3) that for each substitution the equation (4) is satisfied. Hence it is an identity. The general case may be similarly treated.

Ex. 1.

$$\begin{aligned}&\begin{bmatrix} r\\ 1\end{bmatrix}-a(1+y)\begin{bmatrix} r\\ 2\end{bmatrix}+a^2(1+y)(1+ay)\begin{bmatrix} r\\ 3\end{bmatrix}\\ &\qquad -a^3(1+y,\,1+a^2y)\begin{bmatrix} r\\ 4\end{bmatrix}+a^4(1+y,\,1+a^3y)\begin{bmatrix} r\\ 5\end{bmatrix}-\ldots\\ &+(-1)^r a^r(1+y,\,1+a^{r-1}y)\equiv(-1)^r\{(1+ay)(1+a^2y)\ldots(1+a^ry)\}\quad(5)\end{aligned}$$

where $(1+y,\,1+a^py)$ denotes the product $\{(1+y)(1+ay)\ldots(1+a^py)\}$ and $\begin{bmatrix} r\\ p\end{bmatrix}$ denotes $\{(a^r-1)(a^{r-1}-1)\ldots(a^p-1)\}$.

This identity may be proved by substituting ay for y and $-a$ for δ in (1).

Ex. 2.[1]

$$\begin{aligned}&\{(1+a^{2r-1}y)(1+a^{2r+1}y)\ldots(1+a^{4r-3}y)\}\\ &\quad\equiv{}^{2r}S_{2r}\{(1+y)(1+a^2y)(1+a^4y)\ldots(1+a^{2r-2}y)\}\\ &\quad-{}^{2r}S_{2r-2}(a-1)\{(1+a^2y)(1+a^4y)\ldots(1+a^{2r-2}y)\}\\ &\quad+{}^{2r}S_{2r-4}(a^3-1)(a-1)\{(1+a^4y)(1+a^6y)\ldots(1+a^{2r-2}y)\}-\ldots\\ &\quad+(-1)^k\{(a^{2k-1}-1)(a^{2k-3}-1)\ldots(a-1)\}\\ &\quad\times{}^{2r}S_{2r-2k}\{(1+a^{2k}y)(1+a^{2k+2}y)\ldots(1+a^{2r-2}y)\}+\ldots\\ &\quad+(-1)^r\{(a^{2r-1}-1)(a^{2r-3}-1)\ldots(a-1)\}.\qquad\ldots\quad(6)\end{aligned}$$

---

[1] *cf.* "On the Evaluation of Some Factorable Continuants," Art 2, *Bul. Cal. Math. Soc.*, Vol. XIII. In subsequent references this paper will be called Paper 2.

Proof. The $k+1$th term in the right-hand-side expression of (1) is

$$=\{(1+a^{k-1}\delta)(1+a^{k-2}\delta)\ldots(1+\delta)\}\frac{\{(a^{r}-1)(a^{r-1}-1)\ldots(a^{k+1}-1)\}}{\{(a^{r-k}-1)(a^{r-k-1}-1)\ldots(a-1)\}}$$

$$\times\{(y-\delta)(ay-\delta)\ldots(a^{r-k-1}y-\delta)\},$$ by Art 6, Paper 3.

Put

$a=\frac{1}{b^2}$, $y=b^{4r-3}z$ and $\delta=-b^{2r-1}$ and let ${}^{n}\mathrm{B}_r$ denote the sum of the products of n factors $1, b, b^2, \ldots b^{n-1}$ taken r of them at a time. Then the $k+1$th term becomes

$$(-1)^k\{(b^{2k-1}-1)(b^{2k-3}-1)\ldots(b^3-1)(b-1)\}$$

$$\times\frac{\{(b^{2r}-1)(b^{2r-1}-1)\ldots(b^{2r-2k+1}-1)\}}{\{(b^{2k}-1)(b^{2k-1}-1)\ldots(b^2-1)(b-1)}$$

$$\times\,{}^{2r-2k}\mathrm{B}_{2r-2k}\{(1+b^{2k}z)(1+b^{2k+2}z)\ldots(1+b^{2r-2}z)\}$$

$$=(-1)^k\{b^{2k-1}-1)(b^{2k-3}-1)\ldots(b^3-1)(b-1)\}$$

$$\times\,{}^{2r}\mathrm{B}_{2r-2k}\{(1+b^{2k}z)(1+b^{2k+2}z)\ldots(1+b^{2r-2}z)\},$$ by Art 6, Paper 3.

Hence the identity is proved.

Ex. 3.[1]

$$(1+a^{2r-1}y)(1+a^{2r+1}y)\ldots(1+a^{4r-5}y)\}$$

$$\equiv{}^{2r-1}\mathrm{S}_{2r-1}\{(1+y)(1+a^2y)\ldots(1+a^{2r-4}y)\}$$

$$-{}^{2r-1}\mathrm{S}_{2r-3}(a-1)\{(1+a^2y)(1+a^4y)\ldots(1+a^{2r-4})\}+\ldots$$

$$+(-1)^k\,{}^{2r-1}\mathrm{S}_{2r-2k-1}\{(a^{2k-1}-1)(a^{2k-3}-1)\ldots(a-1)\}$$

$$\times\{(1+a^{2k}y)(1+a^{2k+2}y)\ldots(1+a^{2r-4}y)\}+\ldots$$

$$+(-1)^{r-1}\,{}^{2r-1}\mathrm{S}_1\{(a^{2r-3}-1)(a^{2r-5}-1)\ldots(a-1)\} \quad \ldots \quad (7)$$

This may be proved in the same manner as *Ex.* 2 by putting

$a=\frac{1}{b^2}$, $y=b^{4r-5}z$ and $\delta=-b^{2r-1}$ in (1).

---

[1] *cf.* Art 3, Paper 2.

2.[1]

$$\frac{1}{\begin{bmatrix} n \\ 1 \end{bmatrix}} - \frac{1}{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \frac{1}{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \cdot \frac{1+ay}{1+y} - \ldots$$

$$\equiv 0 \text{ or } (-1)^{\frac{n}{2}} \frac{\{(a-1)(a^3-1)\ldots(a^{n-1}-1)\}}{\begin{bmatrix} n \\ 1 \end{bmatrix}\{(1+y)(1+a^2y)\ldots(1+a^{n-2}y)\}}$$

according as n is odd or even; the last term of the series is

$$(-1)^n \frac{\{(1+a^ny)(1+a^{n+2}y)\ldots(1+a^{2n-3}y)\}}{\begin{bmatrix} n \\ 1 \end{bmatrix}\{(1+y)(1+a^2y)\ldots(1+a^{n-3}y)\}}$$

or $$(-1)^n \frac{\{(1+a^{n-1}y)(1+a^{n+1}y)\cdots(1+a^{2n-3}y)\}}{\begin{bmatrix} n \\ 1 \end{bmatrix}\{(1+y)(1+a^2y)\ldots(1+a^{n-2}y)\}} \quad \ldots \quad (8)$$

according as n is odd or even.

Proof. Let ${}^n\beta_r$ denote the series

$$\frac{{}^nS_r}{{}^rS_r} - \frac{{}^nS_{r+1}\,{}^{r+1}S_1}{{}^{r+1}S_{r+1}} + \frac{{}^nS_{r+2}\,{}^{r+2}S_2}{{}^{r+2}S_{r+2}} - \ldots$$

$$+(-1)^k \frac{{}^nS_{r+k}\,{}^{r+k}S_k}{{}^{r+k}S_{r+k}} + \ldots + (-1)^{n-r} \frac{{}^nS_n\,{}^nS_{n-r}}{{}^nS_n}.$$

Since it can be shown, by Art 6, Paper 3, that

$$\frac{{}^nS_{r+k}\,{}^{r+k}S_k}{{}^{r+k}S_{r+k}} = \frac{{}^nS_r}{{}^rS_r} \cdot {}^{n-r}S_k,$$

we have

$${}^n\beta_r = \frac{{}^nS_r}{{}^rS_r}\{1 - {}^{n-r}S_1 + {}^{n-r}S_2 - \ldots + (-1)^{n-r}\,{}^{n-r}S_{n-r}\} = 0, \text{ by } (3)$$

$$\therefore\ {}^n\beta_0 = {}^n\beta_1 = \ldots = {}^n\beta_{n-1} = 0, \text{ but } {}^n\beta_n = \frac{{}^nS_n}{{}^nS_n} = 1.^{2} \quad \ldots \quad (9)$$

---

[1] *cf.* Art 4, Paper 2.

[2] *cf.* Theorem (8), Paper 2.

Now let us take the particular case of (8) when $n=7$, an odd number, then the series is

$$\frac{1}{\begin{bmatrix}7\\1\end{bmatrix}}-\frac{1}{\begin{bmatrix}6\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}+\frac{1}{\begin{bmatrix}5\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}}\cdot\frac{1+ay}{1+y}$$

$$-\frac{1}{\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}}\cdot\frac{1+a^3y}{1+y}+\frac{1}{\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}}\cdot\frac{(1+a^3y)(1+a^5y)}{(1+y)(1+a^2y)}$$

$$-\frac{1}{\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}5\\1\end{bmatrix}}\cdot\frac{(1+a^5y)(1+a^7y)}{(1+y)(1+a^2y)}$$

$$+\frac{1}{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}6\\1\end{bmatrix}}\cdot\frac{(1+a^5y)(1+a^7y)(1+a^9y)}{(1+y)(1+a^2y)(1+a^4y)}$$

$$-\frac{1}{\begin{bmatrix}7\\1\end{bmatrix}}\cdot\frac{(1+a^7y)(1+a^9y)(1+a^{11}y)}{(1+y)(1+a^2y)(1+a^4y)}.$$

If we take the denominator of the last term as the common denominator, then the numerator becomes

$$(1+y)(1+a^2y)(1+a^4y)-\frac{{}^7S_1}{{}^1S_1}(1+y)(1+a^2y)(1+a^4y)$$

$$+\frac{{}^7S_2}{{}^2S_2}(1+ay)(1+a^2y)(1+a^4y)-\ldots$$

$$+\frac{{}^7S_6}{{}^6S_6}(1+a^5y)(1+a^7y)(1+a^9y)-\frac{{}^7S_7}{{}^7S_7}(1+a^7y)(1+a^9y)(1+a^{11}y).$$

If $u_1=(1+y)(1+a^2y)(1+a^4y)$, $u_2=(1+a^3y)(1+a^4y)$, and $u_3=1+a^4y$ then applying (6) to odd terms and (7) to even terms, we can show that the numerator

$$={}^7\beta_0u_1-{}^7\beta_2u_2(a-1)+{}^7\beta_4u_3(a^3-1)(a-1)$$
$$-{}^7\beta_6(a^5-1)(a^3-1)(a-1)=0, \text{ by } (9).$$

Thus the series vanishes.

If $n=6$ an even number, we can similarly show that the numerator is

$${}^6\beta_0 u_1 - {}^6\beta_2 u_2(a-1) + {}^6\beta_4 u_3(a^3-1)(a-1) - (a^5-1)(a^3-1)(a-1) = -(a^5-1)(a^3-1)(a-1).$$

$$\therefore \text{ The series } = -\frac{(a^5-1)(a^3-1)(a-1)}{\begin{bmatrix} 6 \\ 1 \end{bmatrix}(1+y)(1+a^2y)(1+a^4y)}.$$

The general case may be similarly treated.

3. The continuant

$$\left|\begin{array}{lllll} x-\dfrac{a^{n+1}-1}{a^2-1}, & -1, & & & \\ \dfrac{(a^{n+1}-1)(a^{n-1}-1)a^2}{(a^3-1)(a-1)(1+a)^2}, & x-\dfrac{(a^n+1)a}{(1+a)(1+a^2)}, & -1, & & \\ & \dfrac{(a^{n+2}-1)(a^{n-2}-1)a^5}{(a^5-1)(a^3-1)(1+a^2)^2}, & x-\dfrac{(a^n+1)a^2}{(1+a^2)(1+a^3)}, & -1, & \\ \cdots & \cdots & \cdots & & \\ & & & \dfrac{(a^{2n-1}-1)(a-1)a^{3n-4}}{(a^{2n-1}-1)(a^{2n-3}-1)(1+a^{n-1})^2} & x-\dfrac{(a^n+1)a^{n-1}}{(1+a^{n-1})(1+a^n)} \end{array}\right|_n$$

$$\equiv\{(x-1)(x-a^2)\ldots(x-a)(x-a^{n-1})\} \quad \ldots \quad (10)$$

Proof. In evaluating this continuant we are to apply two algebraic relations *viz.*,

$$\frac{(a^{n+r}-1)}{(a^{2r+1}-1)(1+a^{r})}+\frac{(a^{n}+1)a^{r}}{(1+a^{r})(1+a^{r+1})}$$

$$-\frac{(a^{n-r-1}-1)a^{3r+2}}{(a^{2r+1}-1)(1+a^{r+1})}\equiv 1 \qquad \ldots \quad (11)$$

and $$\frac{(a^{n+r-k+2}-1)(a^{n+r-k+1}-1)(a^{n-r}-1)}{(a^{2r+1}-1)(1+a^{r})(a^{k-1}-1)}$$

$$+\frac{(a^{n-r-k+1}-1)(a^{n+r-k+2}-1)(a^{n}+1)a^{r}}{(a^{k-1}-1)(1+a^{r})(1+a^{r+1})}$$

$$-\frac{(a^{n-r-k+1}-1)(a^{n-r-k}-1)(a^{n+r+1}-1)a^{3r+2}}{(a^{2r+1}-1)(1+a^{r+1})(a^{k-1}-1)}$$

$$-(a^{n-k+1}-1)\equiv\frac{a^{k-1}(a^{n-r-k+1}-1)(a^{n+r-k+2}-1)}{(a^{k-1}-1)}\,. \qquad \ldots \quad (12)$$

In the case of (12), if $(1+a^{r})(1+a^{r+1})(a^{2r+1}-1)(a^{k-1}-1)$ be taken as the common denominator of the left-hand-side expression and the factors in each term of the numerator be multiplied together, the numerator will contain eighty terms of which sixty will be cancelled and the remaining twenty are :—

$$-a^{2n-k+2}-a^{2n+r-k+3}-a^{2n+r-k+2}+a^{n-r}+a^{n+1}+a^{2n+3r-k+4}$$

$$+a^{2n+4r-k+4}+a^{2n+3r-k+3}-a^{n+4r+3}-a^{n+3r+3}-a^{n+2r+2}$$

$$-a^{n+4r+2}+a^{n}+a^{n+r+1}+a^{3r+k+1}+a^{3r+k}+a^{4r+k+1}$$

$$-a^{k-1}-a^{r+k}-a^{r+k-1}$$

$$=a^{k-1}(1+a^{r})(1+a^{r+1})(a^{2r+1}-1)(a^{n-r-k+1}-1)(a^{n+r-k+2}-1).$$

Hence the theorem is proved. If we multiply both sides of (12) by $a^{k-1}-1$ and then put $k=1$, we get the theorem (11).

Now let us consider the particular case of the continuant when $n=4$ *viz.*,

$$\left|\begin{array}{cccc} x-\dfrac{a^5-1}{a^3-1}, & -1 & & \\ \dfrac{(a^5-1)(a^3-1)a^2}{(a^3-1)(a-1)(1+a)^2}, & x-\dfrac{(a^4+1)a}{(1+a)(1+a^2)}, & -1, & \\ & \dfrac{(a^6-1)(a^2-1)a^5}{(a^5-1)(a^3-1)(1+a^2)^2}, & x-\dfrac{(a^4+1)a^2}{(1+a^2)(1+a^3)}, & -1 \\ & & \dfrac{(a^7-1)(a-1)a^8}{(a^7-1)(a^5-1)(1+a^3)^2}, & x-\dfrac{(a^4+1)a^3}{(1+a^3)(1+a^4)} \end{array}\right|$$

On this perform the operations :—

$$-\frac{(a^3-1)(a^2-1)(a-1)a^{16}}{(a^5-1)(a^3-1)(a-1)(1+a^2)(1+a^3)}\text{col}_4+\frac{(a^3-1)(a^2-1)a^7}{(a^3-1)(a-1)(1+a^2)}\text{col}_3-\frac{(a^3-1)a^2}{a-1}\text{col}_2+(a+1)\text{col}_1=\overset{(1)}{\text{col}_4};$$

$$\frac{1}{x-1}\overset{(1)}{\text{col}_4}+\frac{(a^2-1)(a-1)(a^6-1)a^7}{(a^3-1)(a-1)(1+a^2)(a-1)}\text{col}_3-\frac{(a^2-1)(a^5-1)a^2}{(a-1)^2}\text{col}_2+\frac{(a+1)(a^4-1)}{a-1}\text{col}_1=\overset{(2)}{\text{col}_4};$$

$$\frac{1}{x-a}\overset{(2)}{\text{col}_4}+0\,\text{col}_3-\frac{(a-1)(a^5-1)(a^4-1)a^2}{(a-1)^2(a^2-1)}\text{col}_2+\frac{(a^4-1)(a^3-1)}{(a-1)^2}\text{col}_1=\overset{(3)}{\text{col}_4};$$

$$\frac{1}{x-a^3}\overset{(3)}{\text{col}_4}+0\,\text{col}_3+0\,\text{col}_2+\frac{(a^4-1)(a^3-1)(a^2-1)}{(a-1)^3(a^3-1)}\text{col}_1=\overset{(4)}{\text{col}_4}.$$

Then we shall find that all the elements, except the first, of the last column, vanish and hence the continuant is evaluated. Further when the elements of the last column, that result from any of these operations, kth for instance, are obtained in the simplest forms by the use of (11) and (12), they will contain $x-a^{k-1}$ as a common factor while the other factors will be the multipliers themselves, the multiplier of rth column occurring in rth element. As an exception to this rule the last element of the last column is always zero except in the case of the first operation.

In the general case if m_r denote the multiplier of rth column and l that of the last column, we have

In the first operation

$$m_r=\lambda_r\left[\begin{matrix} n-1 \\ n-r+1 \end{matrix}\right]$$

where

$$\lambda_r=(-1)^{r-1}\frac{(a+1)a^{\frac{1}{2}(r-1)(3r-2)}}{\{(a^{2r-3}-1)(a^{2r-5}-1)..(a-1)\}\{(1+a)(1+a^2)..(1+a^{r-1})\}}$$

and l is governed by the same rule.

In the second operation

$$m_r=\lambda_r\left[\begin{matrix} n-2 \\ n-r \end{matrix}\right]\frac{a^{n+r-1}-1}{a-1} \quad \text{and} \quad l=\frac{1}{x-1}$$

and so on.

In the kth operation

$$m_r=\lambda_r\left[\begin{matrix} n-k \\ n-r-k+2 \end{matrix}\right]\left[\begin{matrix} n+r-1 \\ n+r-k+1 \end{matrix}\right]\Bigg/\left[\begin{matrix} k-1 \\ 1 \end{matrix}\right]$$

and $$l=\frac{1}{x-a^{k-2}}.$$

4. (*i*).

$$\left[\begin{matrix} n-1 \\ n-r+1 \end{matrix}\right]-\left[\begin{matrix} n-2 \\ n-r \end{matrix}\right]^{n+r-1}S_1+\left[\begin{matrix} n-3 \\ n-r-1 \end{matrix}\right]^{n+r-1}S_2-\ldots$$

$$+(-1)^{n-r}\left[\begin{matrix} r-1 \\ 1 \end{matrix}\right]^{n+r-1}S_{n-r}$$

$$\equiv(-1)^{n-r}\left[\begin{matrix} n-1 \\ n-r+1 \end{matrix}\right]a^{\frac{1}{2}(n-r)(n+r-1)} \qquad \ldots (13)$$

Proof. Let us take the series

$$\begin{bmatrix} n-1 \\ n-r+1 \end{bmatrix}, \begin{bmatrix} n-2 \\ n-r \end{bmatrix}, \begin{bmatrix} n-3 \\ n-r-1 \end{bmatrix} \cdots \begin{bmatrix} r-1 \\ 1 \end{bmatrix}, 0, 0, 0, \ldots$$

and obtain from it

$$\Delta_1, \Delta_2, \ldots \Delta_{r-1}, \Delta_r, \Delta_{r+1}, \ldots \Delta_n, \Delta_{n+1} \ldots \Delta_{n+r-2}, \Delta_{n+r-1}$$

the successive orders of differences by using 1, a, a^2 etc., as the multipliers (see Art 5, Paper 3). Then the first term of Δ_{n+r-1} is

$$(-1)^{n-r} a^{\frac{1}{2}(n-r)(n+r-1)} \begin{bmatrix} n-1 \\ n-r+1 \end{bmatrix}.$$

Hence the identity is proved by Art 5(*i*), Paper 3.

$$(ii) \qquad \frac{(a^p-1)a^{n-p}}{a^{n-p+1}-1} + \frac{a^{n+1}-1}{a^p-1} + \frac{(a^{n+1}-1)(a^{n-p-1}-1)a}{(a^{n-p+1}-1)(a^p-1)}$$

$$\equiv \frac{a^n-1}{a-1} \qquad \ldots \quad (14)$$

and

$$\frac{(a^{n-r}-1)(a^{n+r-p+1}-1)(a^{n+r-p}-1)}{(a^{2r+1}-1)(a^r+1)(a^p-1)} - (a^{n-p}-1)a^{n-p-r}$$

$$- \frac{(a^n+1)(a^{n+r-p+1}-1)(a^{n-p-r}-1)}{(1+a^r)(1+a^{r+1})(a^p-1)}$$

$$- \frac{(a^{n+r+1}-1)(a^{n-p-r}-1)(a^{n-p-r-1}-1)a^{r+1}}{(a^{2r+1}-1)(a^{r+1}+1)(a^p-1)} \equiv 0 \qquad \ldots \quad (15)$$

These two theorems may be proved in the same manner as the theorem (12). From (15) it is clear that if in the lelft-hand-side expression of (12), (−1) be 'substituted for a' in the second term, a^{r+1} for a^{3r+2} in the third term and the fourth term be multiplied by $a^{n-r-k+1}$, the expression vanishes,

5. The operations given in Art 3, may be stated thus :—

$$\lambda_n\begin{bmatrix} n-1 \\ 1 \end{bmatrix}\mathrm{col}_n+\dots+\lambda_r\begin{bmatrix} n-1 \\ n-r+1 \end{bmatrix}\mathrm{col}_r+\dots=\mathrm{col}_n^{(1)}\ ;$$

$$\mathrm{col}_n^{(1)}+\dots+\lambda_r\begin{bmatrix} n-2 \\ n-r \end{bmatrix}\frac{a^{n+r-1}-1}{a-1}\ (x-1)\mathrm{col}_r+\dots=\mathrm{col}_n^{(2)}\ ;$$

$$\dots \qquad \dots \qquad \dots$$

$$\mathrm{col}_n^{(n-r)}\ +0\,\mathrm{col}_{n-1}+0\,\mathrm{col}_{n-2}+\dots+0\,\mathrm{col}_{r+1}$$

$$+\lambda_r\begin{bmatrix} r-1 \\ 1 \end{bmatrix}\begin{bmatrix} n+r-1 \\ 2r \end{bmatrix}\{(x-1)(x-a)\dots(x-a^{n-r-1})\}\Big/\begin{bmatrix} n-r \\ 1 \end{bmatrix}\mathrm{col}_r$$

$$+\dots=\mathrm{col}_n^{(n-r+1)}\ ;$$

$$\mathrm{col}_n^{(n-r+1)}\ +0\,\mathrm{col}_{n-1}+\dots+0\,\mathrm{col}_{r+1}+0\,\mathrm{col}_n+\dots=\mathrm{col}_n^{(n-r+2)}\ ;$$

$$\dots \qquad \dots \qquad \dots$$

We may substitute for the above operations, a single operation in which m_r the multiplier of the rth column will be[1]

$$\lambda_r\Big\{\begin{bmatrix} n-1 \\ n-r+1 \end{bmatrix}+\begin{bmatrix} n-2 \\ n-r \end{bmatrix}\frac{a^{n+r-1}}{a-1}(x-1)$$

$$+\begin{bmatrix} n-3 \\ n-r-1 \end{bmatrix}\begin{bmatrix} n+r-1 \\ n+r-2 \end{bmatrix}(x-1)(x-a)\Big/\begin{bmatrix} 2 \\ 1 \end{bmatrix}+\dots$$

$$+\begin{bmatrix} n-k \\ n-r-k+2 \end{bmatrix}\begin{bmatrix} n+r-1 \\ n+r-k+1 \end{bmatrix}\{(x-1)(x-a)\dots(x-a^{k-2})\}\Big/\begin{bmatrix} k-1 \\ 1 \end{bmatrix}+$$

$$\dots+\begin{bmatrix} r-1 \\ 1 \end{bmatrix}\begin{bmatrix} n+r-1 \\ 2r \end{bmatrix}\{x-1)(x-a)\dots(x-a^{n-r-1})\}\Big/\begin{bmatrix} n-r \\ 1 \end{bmatrix}\Big\}$$

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[1] See Paper 2, Art 6, p. 78.

Thus in m_r the highest power of x is $n-r$ and the co-efficient of x^p

$$=\lambda_r \left\{ \begin{bmatrix} n-p-1 \\ n-r-p+1 \end{bmatrix} {}^{n+r-1}S_p / {}^{p}S_p \right.$$

$$-\begin{bmatrix} n-p-2 \\ n-r-p \end{bmatrix} {}^{n+r-1}S_{p+1} \, {}^{p+1}S_1 / {}^{p+1}S_{p+1}$$

$$+\begin{bmatrix} n-p-3 \\ n-r-p-1 \end{bmatrix} {}^{n+r-1}S_{p+2} \, {}^{p+2}S_2 / {}^{p+2}S_{p+2} - \dots$$

$$\left. +(-1)^{n-r-p} \begin{bmatrix} r-1 \\ 1 \end{bmatrix} {}^{n+r-1}S_{n-r} \, {}^{n-r}S_{n-r-p} / {}^{n-r}S_{n-r} \right\}$$

$$=\lambda_r \frac{{}^{n+r-1}S_p}{{}^{p}S_p} \left\{ \begin{bmatrix} n-p-1 \\ n-r-p+1 \end{bmatrix} - \begin{bmatrix} n-p-2 \\ n-r-p \end{bmatrix} {}^{n+r-p-1}S_1 \right.$$

$$+\begin{bmatrix} n-p-3 \\ n-r-p-1 \end{bmatrix} {}^{n+r-p-1}S_2 - \dots$$

$$\left. +(-1)^{n-r-p} \begin{bmatrix} r-1 \\ 1 \end{bmatrix} {}^{n+r-p-1}S_{n-r-p} \right\}$$

$$=\lambda_r \frac{{}^{n+r-1}S_p}{{}^{p}S_p} (-1)^{n-p-r} \begin{bmatrix} n-p-1 \\ n-p-r+1 \end{bmatrix} a^{\frac{1}{2}(n-p-r)(n-p+r-1)} \text{ by (13)}$$

Substituting the value of λ_r we have the co-efficient of x^p

$$=(-1)^{n-p-1} \frac{\begin{bmatrix} n+r-1 \\ n+r-p \end{bmatrix}}{\begin{bmatrix} p \\ 1 \end{bmatrix}} \begin{bmatrix} n-p-1 \\ n-p-r+1 \end{bmatrix} a^{\frac{1}{2}(n-p)(n-p-1)+(r-1)^2}$$

$$\times \frac{(a+1)\{(a-1)(a^2-1)\dots(a^{r-1}-1)\}}{\{(a^{2r-2}-1)(a^{2r-3}-1)\dots(a^2-1)(a-1)\}}$$

$$=(-1)^{n-p-1} \frac{{}^{n+r-1}S_p \, {}^{n-p-1}S_{r-1}(a+1)}{{}^{2r-2}S_{r-1}} a^{\frac{1}{2}(n-2p)(n-1)+(r-1)^2}.$$

Since n is the order of the continuant, we may divide each multiplier by

$$(-1)^{n-1}(a+1)a^{\frac{1}{2}n(n-1)}$$

and take the coefficient of x^p in m_r as

$$(-1)^p \frac{{}^{n+r-1}S_p\,{}^{n-p-1}S_{r-1}}{{}^{2r-2}S_{r-1}} \cdot \frac{a^{(r-1)^2}}{a^{p(n-1)}}$$

$$\therefore m_r = \frac{a^{(r-1)^2}}{{}^{2r-2}S_{r-1}} \left\{ {}^{n-1}S_{r-1} - \frac{1}{a^{n-1}}\,{}^{n-2}S_{r-1}\,{}^{n+r-1}S_1 x \right.$$

$$+ \frac{1}{a^{2(n-1)}}\,{}^{n-3}S_{r-1}\,{}^{n+r-1}S_2 x^2 - \dots$$

$$+ (-1)^{n-r-1} \frac{1}{a^{(n-r-1)(n-1)}}\,{}^{r}S_{r-1}\,{}^{n+r-1}S_{n-r-1} x^{n-r-1}$$

$$\left. + (-1)^{n-r} \frac{1}{a^{(n-r)(n-1)}}\,{}^{r-1}S_{r-1}\,{}^{n+r-1}S_{n-r} x^{n-r} \right\} \quad \dots \quad (16)$$

The multiplier of the last column

$$= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \lambda_n \div (-1)^{n-1}(a+1)a^{\frac{1}{2}n(n-1)} = a^{(n-1)^2} \frac{{}^{n-1}S_{n-1}}{{}^{2n-2}S_{n-1}}. \qquad (17)$$

Thus this multiplier may also be obtained by (16).

(*i*) Now if the single operation obtained by means of the formula (16) be performed on the continuant of the *n*th order, then from the first row we have

$$\left(x - \frac{a^{n+1}-1}{a^n-1} \right) m_1 - m_2,$$

in which the coefficient of x^p

$$= (-1)^{p-1} \frac{1}{a^{p(n-1)}} \left\{ {}^{n}S_{p-1} a^{n-1} + \frac{a^{n+1}-1}{a^n-1}\,{}^{n}S_p + \frac{{}^{n+1}S_p\,{}^{n-p-1}S_1 a}{{}^{n}S_1} \right\}$$

$$= (-1)^{p-1} a^{\frac{1}{2}p(p-2n+1)} \frac{\begin{bmatrix} n \\ n-p+1 \end{bmatrix}}{\begin{bmatrix} p \\ 1 \end{bmatrix}} \cdot \frac{a^n-1}{a-1} \text{ by (14)}$$

$$= (-1)^{p-1} \frac{1}{a^{\frac{1}{2}n(n-1)}} \cdot \frac{a^n-1}{a-1}\,{}^{n}S_{n-p}.$$

$$\therefore \quad \left(x-\frac{a^{n+1}-1}{a^2-1}\right)m_1-m_2$$

$$=(-1)^{n-1}\frac{1}{a^{\frac{1}{2}n(n-1)}}\;\frac{a^n-1}{a-1}\left\{x^n-{}^nS_1x^{n-1}+{}^nS_2x^{n-2}-\ldots+(-1)^n\,{}^nS_n\right\}$$

$$=(-1)^{n-1}\frac{1}{a^{\frac{1}{2}n(n-1)}}\cdot\frac{a^n-1}{a-1}\left\{(x-1)(x-a)\ldots(x-a^{n-1})\right\} \quad \ldots \quad (18)$$

which is the first element of the last column.

From the $r+1$th row, we get

$$\frac{(a^{n+r}-1)(a^{n-r}-1)a^{2r-1}}{(a^{2r+1}-1)(a^{2r-1}-1)(1+a^r)^2}\,m_r$$

$$+\left\{x-\frac{(a^n+1)a^r}{(1+a^r)(1+a^{r+1})}\right\}m_{r+1}-m_{r+2}$$

in which the coefficient of x^p is

$$=(-1)^p\frac{\begin{bmatrix} n+r \\ n+r-p+2\end{bmatrix}\begin{bmatrix} n-p-1 \\ n-p-r+1\end{bmatrix}a^{2r-1}a^{(r-1)^2}\,{}^pS_p}{\begin{bmatrix} 2r \\ r+1\end{bmatrix}\begin{bmatrix} p-1 \\ 1\end{bmatrix}a^{p(n-1)}}.$$

$$\times\left\{\frac{(a^{n-r}-1)(a^{n+r-p+1}-1)(a^{n+r-p}-1)}{(a^{2r+1}-1)(1+a^r)(a^p-1)}-(a^{n-p}-1)a^{n-p-r}\right.$$

$$-\frac{(a^n+1)(a^{n+r-p+1}-1)(a^{n-p-r}-1)}{(1+a^r)(1+a^{r+1})(a^p-1)}$$

$$\left.-\frac{(a^{n+r+1}-1)(a^{n-p-r-1}-1)(a^{n-p-r}-1)a^{r+1}}{(a^{2r+1}-1)(1+a^{r+1})(a^p-1)}\right\}=0, \text{ by } (15)$$

Thus $(r+1)$th element of the last column is zero. Similarly all the elements, except the first, of the last column vanish.

The product of the elements of the lower minor diagonal

$$=\frac{\begin{bmatrix} n-1 \\ 1\end{bmatrix}^2(a-1)\,a^{2(n-1)^2}}{\begin{bmatrix} 2n-2 \\ 1\end{bmatrix}(a^n-1)^{n-1}S_{n-1}}=\frac{(a-1)a^{2(n-1)^2}}{{}^{2n-2}S_{n-1}(a^n-1)} \quad \ldots \quad (19)$$

Hence the value of the continuant follows readily from (17), (18) and (19).

(*ii*) To illustrate the application of the formula (16), let us consider the continuant of the 5th order, then the single operation is

$$\frac{a^{16}}{{}^8S_4}\ {}^4S_4\mathrm{col}_5+\frac{a^9}{{}^6S_3}\ \{{}^4S_3-\frac{1}{a^4}{}^3S_3\,{}^8S_1x\}\mathrm{col}_4+\ \frac{a^4}{{}^4S_2}\ \{{}^4S_2-\frac{1}{a^4}{}^3S_2\,{}^7S_1x+\frac{1}{a^8}{}^2S_2\,{}^7S_2x^2\}\mathrm{col}_3$$

$$+\frac{a}{{}^3S_1}\ \{{}^4S_1-\frac{1}{a^4}{}^3S_1\,{}^6S_1x+\frac{1}{a^8}{}^2S_1\,{}^6S_2x^2-\frac{1}{a^{12}}\ {}^1S_1\,{}^6S_3x^3\}\mathrm{col}_2+\{1-\frac{1}{a^4}{}^5S_1x+\frac{1}{a^8}{}^5S_2x^2-\frac{1}{a^{12}}\ {}^5S_3x^3+\frac{1}{a^{16}}\ {}^5S_4x^4\}\mathrm{col}_1.$$

On performing this operation we have the continuant

$$=\frac{{}^5S_4}{{}^4S_4a^{16}}\begin{vmatrix} x-\frac{a^6-1}{a^2-1}, & -1, & 0, & 0, & \frac{1}{a^{10}}\cdot\frac{a^5-1}{a-1}\ \{(x-1)(x-a)\ldots(x-a^4)\} \\ \frac{(a^6-1)(a^4-1)a^2}{(a^5-1)(a-1)(1+a)^2}, & x-\frac{(a^6+1)a}{(1+a)(1+a^2)}, & -1, & 0, & 0 \\ & \frac{(a^7-1)(a^3-1)a^5}{(a^5-1)(a^3-1)(1+a^2)^2} & x-\frac{(a^5+1)a^2}{(1+a^2)(1+a^3)}, & -1, & 0 \\ & & \frac{(a^8-1)(a^2-1)a^8}{a^7-1)(a^5-1)(1+a^3)^2}, & x-\frac{(a^5+1)a^3}{(1+a^3)(1+a^4)}, & 0 \\ & & & \frac{(a^9-1)(a-1)a^{11}}{(a^9-1)(a^7-1)(1+a^4)^2}, & 0 \end{vmatrix}$$

$$=(x-1)(x-a)(x-a^2)(x-a^3)(x-a^4).$$

Here the elements of the last column may be obtained by the theorems (14) and (15), taking r equal to 1 less than the number of the row which is considered.

Ex. 1. Since

$$\frac{a^r-1}{a^p-1}=\frac{1+a+a^2+\ldots+a^{r-1}}{1+a+a^2+\ldots+a^{p-1}}=\frac{r}{p} \text{ if } a=1$$

$\therefore$ as a particular case of the continuant of Art 3 when $a=1$, we have

$$\begin{vmatrix} x-\frac{n+1}{2}, & -1, & & & \\ \frac{(n+1)(n-1)}{3\cdot 1\cdot 2^2}, & x-\frac{1}{2}, & -1, & & \\ & \frac{(n+2)(n-2)}{5\cdot 3\cdot 2^2}, & x-\frac{1}{2}, & -1, & \\ \ldots & \ldots & \ldots & \ldots & \\ & & & \frac{(2n-1)\cdot 1}{(2n-1)(2n-3)2^2}, & x-\frac{1}{2} \end{vmatrix}_n$$

$=(x-1)^n$.

6. In the language of Mr. Datta[1] the Heilermann's Theorem is :—If the series

$$\frac{a_0}{x}+\frac{a_1}{x^2}+\frac{a_2}{x^3}+\ldots \qquad \ldots \quad (20)$$

is converted into a continued fraction of the form

$$\frac{a_1}{x+b_1}\,+\,\frac{a_2}{x+b_2}\,+\,\frac{a_3}{x+b_3}\,+\,\ldots \qquad \ldots \quad (21)$$

then the elements of the continued fraction are given by

$$a_r=\frac{k_{r-2}k_r}{k_{r-1}^2} \text{ and } b_r=\frac{{}^1k_{r-1}}{k_{r-1}}-\frac{{}^1k_r}{k_r}$$

where $k_{r+1}=\begin{vmatrix} a_r, & a_{r-1}, & \ldots & a_1, & a_0 \\ a_{r+1}, & a_r, & \ldots & a_2, & a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2r}, & a_{2r-1}, & \ldots & a_{r+1}, & a_r \end{vmatrix}$

and ${}^pk_r$ is obtained from k_{r+1} by deleting the $p+1$th column and the last row. Moreover if $f_r(x)$ and $\phi_{r-1}(x)$ are respectively the denominator and the numerator of the rth convergent then

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[1] Haripada Datta. "On the Failure of Heilermann's Theorem." *Proc. Edin. Math. Soc.*, Vol. 35, 1916-1917.

$$f_r(x)=x^r-\frac{{}^1k_r}{k_r}x^{r-1}+\frac{{}^2k_r}{k_r}x^{r-2}-\ldots+(-1)^r\frac{{}^rk_r}{k_r}$$

and $$\phi_{r-1}(x)=\overset{(r)}{\gamma}_{r-1}x^{r-1}+\overset{(r)}{\gamma}_{r-2}x^{r-2}+\ldots+\overset{(r)}{\gamma}_1 x+\overset{(r)}{\gamma}_0$$

where $$\overset{(r)}{\gamma}_0=a_{r-1}-\frac{{}^1k_r}{k_r}a_{r-2}+\frac{{}^2k_r}{k_r}a_{r-3}-\ldots+(-1)^{r-1}\frac{{}^{r-1}k_r}{k_r}a_0$$

$$\overset{(r)}{\gamma}_1=a_{r-2}-\frac{{}^1k_r}{k_r}a_{r-3}+\ldots+(-1)^{r-2}\frac{{}^{r-2}k_r}{k_r}a_0$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\overset{(r)}{\gamma}_{r-2}=a_1-\frac{{}^1k_r}{k_r}a_0 \qquad \overset{(r)}{\gamma}_{r-1}=a_0$$

The successive convergents to the continued fraction (21) have the property that if the nth convergent is expanded as a power-series in $\frac{1}{x}$, the first $2n$ terms of this expansion will be, term for term, the same as the first $2n$ terms of the series (20) "[1]

If, by the above theorem, the series

$$-\frac{{}^nS_1}{x}+\frac{{}^nS_2}{x^2}-\frac{{}^nS_3}{x^3}+\ldots+(-1)^n\frac{{}^nS_n}{x^n} \qquad \ldots \quad (23)$$

be converted into a continued fraction of the form (21), then the elements of the continued fraction will be given by

$$a_1=\frac{a^n-1}{1-a}\,, \quad b_1=-\frac{a^{n-1}-1}{1-a^n}\,a$$

$$a_r=\frac{(a^{n+r-1}-1)(a^{n-r+1}-1)a^{3r-4}}{(1-a^{2r-1})(1-a^{2r-3})(1+a^{r-1})^2} \text{ and } b_r=-\frac{(a^n+1)a^{r-1}}{(1+a^{r-1})(1+a^r)}$$

Proof. If we expand by division the first convergent $\frac{a_1}{x+b_1}$ as a power series in $\frac{1}{x}$ and equate the first two terms of this expansion with the first two terms of the series (23), we can readily get a_1 and b_1. For other elements we are to find out k_r and ${}^1k_r$.

If $s_p=(-1)^p\,{}^nS_p$ then

$$k_4=\begin{vmatrix} s_4 & s_3 & s_2 & s_1 \\ s_5 & s_4 & s_3 & s_2 \\ s_6 & s_5 & s_4 & s_3 \\ s_7 & s_6 & s_5 & s_4 \end{vmatrix}$$

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[1] For the other part of this theorem, see Datta's paper "On the Theory of Continued Fractions" *Proc. Edin. Math. Soc.*, Vol. 35, 1916-1917.

$$=s_1s_2s_3s_4a^{10}\begin{vmatrix} \dfrac{(a^{n-1}-1)(a^{n-2}-1)(a^{n-3}-1)}{(1-a^2)(1-a^3)(1-a^4)}, & \dfrac{(a^{n-1}-1)(a^{n-2}-1)}{(1-a^2)(1-a^3)}, & \dfrac{a^{n-1}-1}{1-a^2}, & 1 \\ \dfrac{(a^{n-2}-1)(a^{n-3}-1)(a^{n-4}-1)}{(1-a^3)(1-a^4)(1-a^5)}\,a^3, & \dfrac{(a^{n-2}-1)(a^{n-3}-1)}{(1-a^3)(1-a^4)}\,a^2, & \dfrac{a^{n-2}-1}{1-a^3}\,a, & 1 \\ \dfrac{(a^{n-3}-1)(a^{n-4}-1)(a^{n-5}-1)}{(1-a^4)(1-a^5)(1-a^6)}\,a^6, & \dfrac{(a^{n-3}-1)(a^{n-4}-1)}{(1-a^4)(1-a^5)}\,a^4, & \dfrac{a^{n-3}-1}{1-a^4}\,a^2, & 1 \\ \dfrac{(a^{n-4}-1)(a^{n-5}-1)(a^{n-6}-1)}{(1-a^5)(1-a^6)(1-a^7)}\,a^9, & \dfrac{(a^{n-4}-1)(a^{n-5}-1)}{(1-a^5)(1-a^6)}\,a^6, & \dfrac{a^{n-4}-1}{1-a^5}\,a^3, & 1 \end{vmatrix}$$

On this determinant perform the operations $\text{row}_1-\text{row}_2$; $\text{row}_2-\text{row}_3$ and $\text{row}_3-\text{row}_4$ and then by the algebraic relation

$$\frac{a^k-1}{1-a^l}-\frac{a^k-a^m}{1-a^{l+m}}=\frac{(a^{k+l}-1)(1-a^m)}{(1-a^l)(1-a^{l+m})} \qquad \ldots \quad (24)$$

we have $$k_4=s_1s_2s_3s_4a^{13}\frac{(a^{n+1}-1)^3(1-a^3)(1-a^2)(1-a)}{(1-a^2)(1-a^3)^2(1-a^4)^2(1-a^5)}\begin{vmatrix} \dfrac{(a^{n-2}-1)(a^{n-3}-1)}{(1-a^4)(1-a^5)}, & \dfrac{a^{n-2}-1}{1-a^4}, & 1 \\ \dfrac{(a^{n-3}-1)(a^{n-4}-1)}{(1-a^5)(1-a^6)}\,a^2, & \dfrac{a^{n-3}-1}{1-a^5}\,a, & 1 \\ \dfrac{(a^{n-4}-1)(a^{n-5}-1)}{(1-a^6)(1-a^7)}\,a^4, & \dfrac{a^{n-4}-1}{1-a^6}\,a^2, & 1 \end{vmatrix}$$

On this last determinant performing the operations $\text{row}_1-\text{row}_2$ and $\text{row}_2-\text{row}_3$ and applying the theorem (24) we get

$$k_4=s_1s_2s_3s_4a^{14}\,\frac{(a^{n+1}-1)^3(1-a^3)(1-a^2)(1-a)}{(1-a^3)(1-a^6)^2(1-a^4)^2(1-a^5)}\cdot\frac{(a^{n+2}-1)^2(1-a^2)(1-a)}{(1-a^4)(1-a^5)^2(1-a^6)}\begin{vmatrix}\dfrac{a^{n-3}-1}{1-a^6}, & 1\\[2ex] \dfrac{a^{n-4}-1}{1-a^7}a, & 1\end{vmatrix}$$

Then by the operation $\text{row}_1-\text{row}_2$ and the use of (24), we have

$$k_4=\frac{(a^{n+3}-1)(a^{n+2}-1)^2(a^{n+1}-1)^3(a^n-1)^4(a^{n-1}-1)^3(a^{n-2}-1)^2(a^{n-3}-1)}{(1-a)(1-a^2)^2(1-a^3)^3(1-a^4)^4(1-a^5)^3(1-a^6)^2(1-a^7)}({}^4S_4)^4.$$

Generally we have

$$k_r=\frac{\{(a^{n+r-1}-1)(a^{n+r-2}-1)^2(a^{n+r-3}-1)^3\ldots(a^{n+1}-1)^{n-1}(a-1)^r(a^{n-1}-1)^{r-1}\ldots(a^{n-r+2}-1)^2(a^{n-r+1}-1)\}}{\{(1-a)(1-a^2)^2(1-a^3)^3\ldots(1-a^{r-1})^{r-1}(1-a^n)^r(1-a^{r+1})^{r-1}\ldots(1-a^{2r-2})^2(1-a^{2r-1})\}}\;({}^rS_r)^r$$

Similarly ${}^1k_r=$

$$\frac{\{(a^{n+r-1}-1)(a^{n+r-2}-1)^2(a^{n+r-3}-1)^3\ldots(a^{n+1}-1)^{r-1}(a^n-1)^r(a^{n-1}-1)^{r-1}\ldots(a^{n-r+2}-1)^2(a^{n-r+1}-1)\}\{(1-a^{r+1})(1-a^{r+2})\ldots(1-a^{2r-1})\}(a^{n-r}-1)}{\{(1-a^2)(1-a^3)^2\ldots(1-a^r)^{r-1}(1-{}^{r+1})^r(1-a^{r+2})^{r-1}\ldots(1-a^{2r-1})^2(1-a^{2r})\}\{(1-a)(1-a^2)\ldots(1-a^{r-1})\}(1-a)}$$

$$\times\,{}^{r+1}S_{r+1}({}^rS_r)^{r-1}$$

$${}^pk_r=\frac{\{(a^{n+r-1}-1)(a^{n+r-2}-1)^2\ldots(a^n-1)^r(a^{k-1}-1)^{r-1}\ldots(a^{n-r+1}-)\}\{(1-a^{r+1})(1-a^{r+2})\ldots(1-a^{2r-p})\}\{(a^{n-r+p+1}-1)(a^{n-r+p-2}-1)\ldots(a^{n-r}-1)\}}{\{(1-a^2)(1-a^3)^2\ldots(1-a^r)^{r-1}(1-a^{r+1})^r(1-a^{r+2})^{r-1}\ldots(1-a^{2r-1})^2(1-a^{2r})\}\{(1-a)(1-a^2)\;(1-a^{n-p})\}\{(1-a)(1-a^2)\ldots(1-a^p)\}}$$

$$\times\,({}^{r+1}S_{r+1})^p({}^rS_r)^{r-p}$$

Hence
$$\frac{{}^pk_r}{k_r}=\frac{\{(1-a^{p+1})(1-a^{p+2})\ldots(1-a^r)\}\{(a^{n-r+p-1}-1)(a^{n-r+p-2}-1)\ldots(a^{n-r}-1)\}a^{rp}}{\{(1-a^{2r-p+1})(1-a^{2r-p+2})\ldots(1-a^{2r})\}\{1-a)(1-a^2)\ldots(1-a^{r-p})\}}\,{}^1$$

The last equation gives us the successive convergents to the continued fraction.

---

[1] It is to be remembered that the product $\{(a^n-1)(a^{n+d}-1)\ldots(a^r-1)\}$ is to be taken as unity if d is positive but r is less than is n or if d is negative but r is greater than n.

2

ON AN APPLICATION OF BESSEL FUNCTIONS TO PROBABILITY

BY

ABANIBHUSAN DATTA

1. Some remarkable definite integrals involving Bessel Functions have been evaluated by Sonine in his elaborate memoir[1] in the *Math. Annalen*. Nicholson[2] in the *Quarterly Journal* generalised some of Sonine's results.

"A very remarkable advance in the theory of random variations and of flights in two dimensions is due to J. C. Kluyver[3] who has discovered an expression for the probability of various resultants in the form of a definite integral involving Bessel Functions. His exposition is rather concise." His paper has been reproduced with slight changes of notation by Lord Rayleigh[4] who has studied various aspects of the problem and also given the most general result of one of the particular cases.

In a previous paper,[5] I have attempted to evaluate more general forms of some of the integrals given by Sonine. In the first part of the present paper, I have employed the method used in my previous paper to extend the results obtained therein and have given a very general case of the integral involving the product of a number of Bessel Functions. In the second part, I have applied this general form to the problem studied by Kluyver and have obtained by *a completely different method* the result obtained therein.

I am indebted to Dr. S. K. Banerji for directing my attention to Kluyver's result and also to Rai A. C. Bose Bahadur for his valuable suggestions.

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[1] Sonine, *Math. Annalen*, Band 16, page 1.

[2] Nicholson, *Quar. Journal*, Vol. 48, part IV (1920) p. 321.

[3] J. C. Kluyver, *Koninklijke Akademie Van Wetenschappen te Amsterdam*, *Verslag van de gewone vergadingen der Wisen-Naturkundige Afduling*, Deel XIV, 1st Gedeelte, 30 Sept, 1905, pp. 305-24.

[4] Lord Rayleigh, *Scientific Papers*, Vol. VI, page 610.

[5] Datta, *Bull. Cal. Math. Soc.*, Vol. XI, No. 4, p. 221.

2. Sonine has given the elegant formula

$$\int_0^\infty J_m(px)J_m(qx)e^{-hx^2}x\,dx$$

$$=\frac{p^m q^m}{\sqrt{\pi}\;2^m\Gamma(m+\frac{1}{2})}\,\frac{1}{(2h)^{m+1}}\int_{-1}^{1} e^{-\frac{p^2+q^2-2pqt}{4h}}(1-t^2)^{m-\frac{1}{2}}dt$$

$$(m>-\tfrac{1}{2}). \quad \ldots \quad (1)$$

It has been shown by me in my previous paper "On an Extension of Sonine's Integral in Bessel Functions" that by substituting, in the above, for h, $h+\frac{c}{2u}$ and multiplying both sides by $\frac{1}{2\pi}e^{\frac{c^m}{2}}\frac{dr}{u^{m+1}}$ and integrating with respect to r between the limits $-\infty$ and $+\infty$, we have

$$\int_0^\infty J_m(px)J_m(qx)J_m(cx)e^{-hx^2}\frac{dx}{x^{m-1}}$$

$$=\frac{p^m q^m c^m}{\sqrt{\pi}\;2^{2m}\sqrt{\pi}\;\{\Gamma(m+\frac{1}{2})\}^2}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\frac{dt}{(2h)^{m+1}}$$

$$\times\int_{-1}^{1} e^{-\frac{c^2+v^2-2cvz}{4h}}(1-z^2)^{m-\frac{1}{2}}dz$$

$$(m>-\tfrac{1}{2}) \qquad (2)$$

where v^2 is equal to p^2+q^2-2pqt and is a positive quantity.

The analogy between the two equations (1) and (2) suggests that we can employ the method indicated above to obtain the integral of the product of four Bessel Functions.

Thus, substituting for h, $h+\frac{b}{2u}$, and repeating the same process, we can write

$$\int_0^\infty J_m(px)J_m(qx)J_m(cx)J_m(bx)e^{-hx^2}\frac{dx}{x^{2m-1}}$$

$$=\frac{p^m q^m c^m b^m}{\pi 2^{3m}\sqrt{\pi}\,\{\Gamma(m+\frac{1}{2})\}^3}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\frac{dt}{(2h)^{m+1}}$$

$$\times\int_{-1}^{1}(1-z^2)^{m-\frac{1}{2}}dz\int_{-1}^{1}e^{-\frac{b^2+\omega^2-2b\omega y}{4h}}(1-y^2)^{m-\frac{1}{2}}dy$$

where $\omega^2=c^2+v^2-2cvz$ and $m>-\frac{1}{2}$.

Using the same method to obtain the integral of the product of any number of Bessel Functions, we can write as a general form (number of Bessel Functions being n)

$$\int_0^\infty J_m(px)J_m(qx)J_m(cx)J_m(bx)\ldots J_m(kx)e^{-hx^2}\frac{dx}{x^{m(n-2)-1}}$$

$$=\frac{p^m q^m c^m b^m\ldots k^m}{\{\pi\}^{\frac{n-1}{2}}\{\Gamma(m+\frac{1}{2})\}^{\frac{n-1}{2}}2^{m(n-1)}}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt$$

$$\times\int_{-1}^{1}\ldots\int_{-1}^{1}\ldots\int_{-1}^{1}e^{-\frac{k^2+u^2-2uka}{4h}}(1-a^2)^{m-\frac{1}{2}}da$$

$$[m>-\tfrac{1}{2}].$$

Now substituting for h, $h=\frac{l}{2u}$, and multiplying both sides by

$$\frac{1}{2\pi}e^{\frac{lu}{2}-\frac{l\gamma^2}{2u}}\frac{dr}{u^{n+1}}$$

and integrating with regard to r between the limits $-\infty$ and $+\infty$, we have

$$\int_0^\infty J_m(px)J_m(qx)J_m(cx)\ldots J_m(kx)\ \frac{J_n(l\sqrt{x^2+\gamma^2})\,dx}{x^{m(n-1)-1}\ (\sqrt{x^2+\gamma^2})^n}$$

$$=\frac{p^m q^m c^m \cdots k^m l^{-n}}{\{\pi\}^{\frac{n-1}{2}}\{\Gamma(m+\frac{1}{2})\}^{\frac{n-1}{2}}2^{m(n-1)}}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt\iiint\ldots$$

$$\times\int_\beta^1 (1-a^2)^{m-\frac{1}{2}}J_{n-m-1}(\gamma\sqrt{l^2-k^2-u^2+2uka})$$

$$\times(\sqrt{l^2-k^2-u^2+2uka})^{n-m-1}\gamma^{m-n+1}da.$$

where $\beta=1$, for $l^2<(k-u)^2$;

$$\beta=\frac{k^2+u^2-l^2}{2ku},\text{ for }(k-u)^2<l^2<(k+u)^2\ ;$$

$$\beta=-1,\text{ for }l^2>(k+u)^2.$$

For $\gamma=0$, we obtain

$$\int_0^\infty J_m(px)J_m(qx)\ \ldots\ \frac{J_m(kx)J_n(lx)}{x^{m(n-1)+(n-1)}}\ dx$$

$$=\frac{p^m q^m\ldots k^m l^{-n}}{\{\sqrt{\pi}\}^{\frac{n-1}{2}}\{(m+\frac{1}{2})\}^{\frac{n-1}{2}}2^{m(n-1)}}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt$$

$$\times\int_{-1}^{1}\ldots\int_{\beta}^{1}(1-a^2)^{m-\frac{1}{2}}(\sqrt{l^2-k^2-u^2+2uka})^{n-m-1}da.$$

$$n>m>-\tfrac{1}{2}.$$

Now if $n=m+1$

$$\int_0^1 J_m(px)J_m(qx)\ldots J_m(kx)\frac{J_{m+1}(lx)}{x^{m^2+m}}\,dx$$

$$=\frac{p^m q^m\ldots k^m l^{-m-1}}{\{\sqrt{\pi}\}^{\frac{n-1}{2}}\{\Gamma(m+\frac{1}{2})\}^{\frac{n-1}{2}}2^{mm}}$$

$$\int_{-1}^1 (1-t^2)^{m-\frac{1}{2}}dt\ldots\int_\beta^1 (1-a^2)^{m-\frac{1}{2}}da.$$

Now if $m=0$,

$$\int_{-1}^1 J_0(px)J_0(qx)\ldots J_0(kx)J_1(lx)dx$$

$$=\frac{1}{l\pi^{n-1}}\int_{-1}^1 (1-t^2)^{-\frac{1}{2}}dt\ldots\int_\beta^1 (1-a^2)^{m-\frac{1}{2}}da.$$

If in this integral, we put

$$t=\cos\theta_1,\ldots a=\cos\theta_{n-1},$$

we get (the number of Bessel Functions being $n+1$)

$$l\int_0^\infty J_0(px)J_0(qx)\ldots J_0(kx)J_1(lx)dx$$

$$=\left(\frac{2}{\pi}\right)^{n-1}\int_0^{\frac{\pi}{2}} d\theta_1\ldots\int_0^{\frac{\pi}{2}} d\theta_{n-1}. \qquad \ldots \quad (1)$$

3. It would appear from the above that there are certain relations between $p, q, c \cdots k, l$; $v, \omega \ldots u$ and $t, z, \ldots a$. These may be interpreted geometrically thus

$$
\begin{aligned}
\text{Since}\quad v_2 &= p^2+q^2-2pqt \quad \text{where } t=\cos\theta_1\\
\omega^2 &= c^2+v^2-2cvz \qquad ,, \quad z=\cos\theta_2\\
\ldots &\qquad \ldots \qquad \ldots\\
l^2 &= t^2+u^2-2uka \qquad ,, \quad a=\cos\theta_{n-1},
\end{aligned}
$$

it is clear that t is the cosine of angle of the triangle having p, v and q for its sides included between the sides p and q ; z is the cosine of the angle of the triangle having c, v and b for its sides included between the sides c and v and so on. Hence it is obvious that $p, q, c, \ldots k, l$ form a polygon having $v, \omega \ldots u$ as the successive diagonals joining one of the angular points to all others in succession. Hence it is evident from the above geometrical interpretation that the integral on the left hand side is equal to the multiple integral on the right hand side or zero according as

$$
\begin{aligned}
v^2 &= p^2+q^2-2pqt \text{ or not}\\
\omega^2 &= c^2+v^2-2cvz \text{ or not}\\
l^2 &= k^2+u^2-2uka \text{ or not}
\end{aligned}
$$

i.e., according as we can form a triangle having p, q, and v for its sides or not, because only in the former case v is a real positive quantity ; according as we can form a triangle having c, v and ω as its sides or not, because the integral is not equal to zero when ω is a real positive quantity and that is only possible when ω is the third side of the triangle having c, ω, and v for its sides ; and so on ; and hence (combining the above conditions) we see that according as we can form a polygon having $p, q, c \ldots k, l$ as its sides, $v, \omega, \ldots u$ being the successive diagonals joining one of the vertices to all others.

4. "We are now in a position to investigate the probability $P_n(rl_1, l_2 \ldots l_n)$ that after n stretches l_1, $l_2 \ldots l_n$ taken in directions at random, the distance from the stretching point 0 shall be less than an assigned magnitude r. The direction of the first stretch l is plainly a matter of indifference. On the other hand, the probability that the angles θ lie within the limits θ_1 and $\theta_1+d\theta_1$, θ_2 and $\theta_2+d\theta_2$ $\ldots \theta_{n-1}$ and $\theta_{n-1}+d\theta_{n-1}$ is $\frac{1}{(\pi)^{n-1}} d\theta_1 d\theta_2 \ldots d\theta_{n-1}$ which is now to be integrated under the conditions that the nth radius vector shall be less than r."

We have shown in the previous articles that

$$\int_0^\infty J_0(px)J_0(qx)\ldots dx$$

$$=\left(\frac{2}{\pi}\right)^{n-1}\int_0^{\pi/2} d\theta_1 \int_0^{\frac{\pi}{2}} d\theta_2 \ldots \int_0^{\frac{\pi}{2}} d\theta_n \text{ or zero}$$

according as we can form a polygon having $p, q,\ldots$ for its sides and $v, \omega\ldots$ as its successive diagonals or not. Hence the probability that the $(n+1)$th radius vector after $(n+1)$ stretches shall be less than an assigned magnitude is

$$P_n(p,q\ldots)=\int_0^\infty J_0(px)\ldots dx$$

[number of Bessel functions being $n+1$].

3

ON VORTEX RINGS OF FINITE CIRCULAR SECTION IN INCOMPRESSIBLE FLUIDS

BY

NRIPENDRANATH SEN

Introduction.

1. In a recent issue [1] of the *Bulletin of the Calcutta Mathematical Society*, it has been shewn that when the vorticity at any point of a moving circular vortex ring of finite section varies as the n^{th} power of the distance of the point from the axis of the ring, its cross-section does not remain circular but gets elongated in the direction of its motion of translation. Although the steady motion of vortex rings has attracted considerable attention of many eminent mathematicians including Kelvin [2], Hicks [3], Chree [4], Basset [5], Dyson [6], Thomson [7] and others, no previous writer has attempted the problem of the motion of vortex rings of finite circular section.

In the present paper, I have shown that for a certain law of vorticity, it is possible for a ring to move with invariable circular section. The law of vorticity and the velocity of translation have been calculated for fairly thick rings. It has been found that to a certain approximation the velocity of translation is identical with that of a ring with constant vorticity, this being due to the fact that correct to that order of approximation the vorticity may be supposed to be constant over the cross section of the ring.

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[1] Nripendranath Sen—"On Circular Vortex Rings of Finite Section in Incompressible Fluids" *Bull. Cal. Math. Soc.*, Vol. 13, p. 117, 1922.

[2] Kelvin—"*Collected Scientific papers*," Vol. 4, p. 67.

[3] Hicks—"*Phil. Trans A*, Vol. 175, 1884; also Vol. 176, 1885.

[4] Chree—"*Proc. Edin. Math. Soc.*," Vol. 6, 1888.

[5] Basset—Hydrodynamics, part II.

[6] Dyson—"Potential of Anchor Ring," parts I and II. *Phil. Trans A*, Vol. 184, 1893.

[7] Thomson—"Motion of Vortex Rings."
Also Gray—"Notes on Hydrodynamics," *Phil. Mag.* (6), Vol. 28, p. 13, 1914.
Lamb—"Hydrodynamics," *Ed.* IV, 1916.

2. Let 2ω = vorticity, k = strength of the vortex

c = radius of the "circular axis"

ρ, ϕ, z = cylindrical co-ordinates of any point referred to the centre of the circular axis as origin and the axis of the ring as z-axis.

r = distance of any point from the circular axis

θ = inclination of this distance to the plane of the "circular axis," so that $\rho = c - r\cos\theta$.

V = velocity of translation of the ring parallel to z-axis

$$J = \int_0^{\pi} \frac{c\cos\phi \, d\phi}{[z'^2 + c^2 - 2c\rho'\cos\phi + \rho'^2]^{\frac{1}{2}}}$$

a = radius of the cross-section

$$l = \log\frac{8c}{r} - 2, \qquad s = \frac{r}{c},$$

$$\lambda = \log\frac{8c}{a} - 2, \qquad \sigma = \frac{a}{c},$$

$$\nabla^2 = \frac{d^2}{dc^2} + \frac{d^2}{dz'^2}, \quad \frac{d}{dc} = \nabla\cos a, \qquad \frac{d}{dz'} = \nabla\sin a$$

ψ = Stokes' stream function.

Then, it can be proved that at any point (ρ', ϕ', z'), outside the vortex filament[1]

$$\psi = \frac{\rho'}{2\pi}\iiint \frac{\omega\rho\cos\phi \, d\phi \, d\rho dz}{\{(z'-z)^2 + \rho'^2 - 2\rho\rho'\cos\phi + \rho^2\}^{\frac{1}{2}}}$$

$$= \frac{\rho'}{\pi}\iint \omega e^{-x\frac{d}{dc} - z\frac{d}{dz'}} \, dx dz \; J \qquad \ldots \qquad (1)$$

---

[1] *Bull. Cal. Math. Soc.*, Vol. 12, p. 120.

where the integral is to be taken over any circular section of the ring.

Now, let

$$\omega = A_0[1 + A_1 r\cos\theta + A_2 r^2\cos 2\theta + A_3 r^3\cos 3\theta + \cdots \quad \ldots] \qquad \ldots \quad (2)$$

$$\therefore\ k = \int_0^{2\pi}\int_0^{a} 2\omega\, r\, dr\, d\theta = 2\pi a^2 A_0 \qquad \ldots \quad (3)$$

From (1) and (2), we have

$$\psi = \frac{\rho'}{\pi}\int_0^{a}\int_0^{2\pi} e^{-r\nabla\cos(\theta-\alpha)} A_0(1 + A_1 r\cos\theta + A_2 r^2\cos 2\theta + \ldots) r\,dr\,d\theta\,J.$$

$$= \frac{\rho' A_0}{\pi}\int_0^{a}\int_0^{2\pi} \{I_0(r\nabla) - 2I_1(r\nabla)\cos(\theta-\alpha) + 2I_2(r\nabla)\cos 2(\theta-\alpha) + \ldots\}$$

$$\times\{1 + A_1 r\cos\theta + A_2 r^2\cos 2\theta + \ldots \quad \ldots\} r\,dr\,d\theta\,J.$$

where I_n is Bessel Function of the n^{th} order with imaginary modulus.

$$= \frac{k\rho'}{\pi a^2}\left[\frac{a}{\nabla} I_1(a\nabla) - A_1\cos\alpha\frac{a^2}{\nabla} I_2(a\nabla) + A_2\cos 2\alpha\,\frac{a^3}{\nabla} I_3(a\nabla) - \text{etc.}\right] J$$

$$\left[\because \int_0^{a} r^{n+1} I_n(r\nabla)\,dr = \frac{a^{n+1}}{\nabla} I_{n+1}(a\nabla)\right]^1$$

$$= \frac{k\rho'}{2\pi}\left[1 + \frac{a^2\nabla^2}{8} + \frac{a^4\nabla^4}{192} + \frac{a^6\nabla^6}{3072} + \ldots \quad \ldots\right.$$

$$-\frac{A_1}{4}\cos\alpha\,\nabla a^2\left(1 + \frac{a^2\nabla^2}{12} + \frac{a^4\nabla^4}{384} + \ldots \quad \ldots\right)$$

---

[1] Whittaker "Mod. Anal.," p. 366, 17.7.

$$+\frac{A_2 a^4}{24}\nabla^2\cos 2a\left(1+\frac{a^2\nabla^2}{16}+\frac{a^4\nabla^4}{640}+\dots\quad\dots\right)$$

$$-\frac{A_3 a^6}{192}\nabla^3\cos 3a\left(1+\frac{a^2\nabla^2}{20}+\dots\quad\dots\right)\dots\quad\dots\Big]\,J$$

$$=\frac{k\rho'}{2\pi}\Big[1+\left(\frac{a^2}{8}-\frac{A_1a^2c}{4}+\frac{A_2a^4}{24}\right)\frac{1}{c}\,\frac{d}{dc}$$

$$+\Big\{-\frac{a^4}{192}-\frac{A_1a^4c}{48}+A_2a^4\left(\frac{c^2}{12}+\frac{a^2}{384}\right)$$

$$-\frac{3A_3a^6c}{64}\Big\}\left(\frac{1}{c}\,\frac{d}{dc}\right)^2+\Big\{\frac{a^6}{3072}+\frac{A_1a^6c}{1536}$$

$$+\frac{A_2a^6}{192}\left(c^2-\frac{a^2}{80}\right)-\frac{A_3a^6c^3}{48}\left(1+\frac{9\sigma^2}{80}\right)\Big\}\left(\frac{1}{c}\,\frac{d}{dc}\right)^3\text{ etc.}\Big]\,J\quad(4)[1]$$

Now,

$$J\frac{\rho'}{c}=l-\frac{l+1}{2}s\cos\theta+\left(\frac{2l+5}{16}-\frac{l}{16}\cos 2\theta\right)s^2$$

$$+\left(\frac{3l+5}{64}\cos\theta-\frac{3l-1}{192}\cos 3\theta\right)s^3$$

$$+\left(\frac{12l+11}{2048}+\frac{12l+17}{768}\cos 2\theta-\frac{15l-8}{3072}\cos 4\theta\right)s^4+\text{etc.}\quad\dots\quad(5)[2]$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\,\frac{d}{dc}\right)J=\frac{1}{c^2s}\Big\{-\cos\theta+\left(\frac{2l+3}{4}+\frac{\cos 2\theta}{4}\right)s$$

$$+\left(\frac{4l+1}{32}\cos\theta+\frac{\cos 3\theta}{32}\right)s^2+\Big(-\frac{4l+7}{128}+\frac{4l+1}{64}\cos 2\theta$$

$$+\frac{\cos 4\theta}{128}\Big)s^3+\text{etc.}\Big\}\quad\dots\quad(6)$$

---

[1] For simplification, see *Bull. Cal. Math. Soc*, Vol. 13, p. 124.

[2] Dyson—"*Phil, Trans.*," *Ibid*, part I, p. 54; part II, pp. 1086-87.

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=\frac{1}{c^4 s^2}\left\{\cos 2\theta-\frac{\cos\theta+\cos 3\theta}{4}s\right.$$

$$\left.-\left(\frac{12l+9}{32}+\frac{\cos 2\theta}{4}+\frac{\cos 4\theta}{32}\right)s^2+\text{etc.}\right\} \quad \ldots \quad (7)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=-\frac{1}{c^6 s^3}\left\{2\cos 3\theta+\left(\cos 2\theta-\frac{\cos 4\theta}{2}\right)s+\ldots\right\} \quad (8)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{1}{c^8 s^4}\left\{6\cos 4\theta+\ \cdots\right\} \quad \ldots \quad (9)$$

etc., etc., etc.

Hence on the surface of the vortex ring, we have, after substitution and simplification,

$$\psi=\frac{kc}{2\pi}\left[\text{const.}-\left\{\frac{(\lambda+1)}{2}\sigma-\frac{3\lambda+5}{64}\sigma^3\right.\right.$$

$$+\left(\frac{1}{8}-\frac{A_1 c}{4}+\frac{A_2 a^2}{24}\right)\left(1-\frac{\sigma^2}{32}(4\lambda+1)\right)\sigma$$

$$\left.+\frac{\sigma^3}{4}\left(-\frac{1}{192}-\frac{A_1 c}{48}+\frac{A_2 c^2}{12}-\frac{3A_3 a^2 c}{64}\right)\right\}\cos\theta$$

$$-\left\{\frac{\lambda}{16}\sigma^2-\frac{\sigma^2}{4}\left(\frac{1}{8}-\frac{A_1 c}{4}+\frac{A_2 a^2}{24}\right)\right.$$

$$\left.+\left(\frac{1}{192}+\frac{A_1 c}{48}-\frac{A_2 c^2}{12}+\frac{3A_3 a^2 c}{64}\right)\sigma^2\right\}\cos 2\theta$$

$$-\left\{\frac{(3\lambda-1)}{192}-\frac{1}{32}\left(\frac{1}{8}-\frac{A_1 c}{4}+\frac{A_2 a^2}{4}\right)\right.$$

$$-\left(\frac{1}{192}+\frac{A_1 c}{48}-\frac{A_2 c^2}{12}+\frac{3A_3 a^2 c}{64}\right)\frac{1}{4}$$

$$\left.\left.+\left(\frac{1}{1536}+\frac{A_1 c}{768}+\frac{A_2 c^2}{96}-\frac{A_3 c^3}{24}\right)+\text{etc.}\right\}\sigma^3\cos 3\theta+\ldots\right] \quad (10)$$

Further, let us suppose that the "centroid" of the vortex filament lies on the "circular axis" of the ring. In that case we must have

$$\int_0^{2\pi}\int_0^{a} \omega r \cos\theta \, r \, dr \, d\theta = 0$$

Hence from (2), we have $A_1 = 0$ ... (11)

Also, from the boundary condition for a velocity of translation V parallel to z—axis, we have

$$\psi = \frac{V\rho^2}{2} + \text{constant on the surface of the ring}$$

$$= \left[\text{constant} - Vac\cos\theta + \frac{Va^2}{4}\cos 2\theta \right].$$

Hence, from this and (10), by equating co-efficients of $\cos\theta$, etc. (always neglecting quantities of the order σ^4 and higher powers of σ), we have

$$\frac{k}{2\pi c}\left\{\frac{\lambda+1}{2} - \frac{3\lambda+5}{64}\sigma^2 + \frac{1}{8} + \frac{A_2 a^2}{24} - \frac{\sigma^2}{256}(4\lambda+1) - \frac{\sigma^2}{4}\left(\frac{1}{192} - \frac{A_2 c^2}{12}\right)\right\} = V \quad \text{... (12)}^1$$

$$\frac{k}{2\pi c}\left\{-\frac{\lambda}{16} + \frac{1}{32} - \frac{1}{192} + \frac{A_2 c^2}{12}\right\} = \frac{V}{4} \quad \text{... (13)}$$

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[1] In obtaining results (12) to (14), $A_2 a^2$, $A_3 a^3$ have been supposed (it will be proved afterwards, see results (16) and (17)) to be of the order σ^2, σ^3 respectively.

$$\frac{3\lambda-1}{192}-\frac{1}{256}-\frac{1}{768}+\frac{A_2c^2}{48}+\frac{1}{1536}+\frac{A_2c^2}{96}-\frac{A_3c^3}{24}=0 \quad \ldots \quad (14)$$

Solving for $V,\ A_2,\ A_3,$

we obtain

$$V=\frac{k}{2\pi c}\left[\frac{4\lambda+5}{8}+\frac{60\lambda+11}{768}\sigma^2\right] \quad \ldots \quad (15)$$

$$A_2c^2=\frac{36\lambda+25}{16}+\frac{60\lambda+11}{256}\sigma^2 \quad \ldots \quad (16)$$

$$A_3c^3=\frac{33\lambda+15}{16}+\frac{180\lambda+33}{1024}\sigma^2 \quad \ldots \quad (17)$$

Since, we neglect terms containing σ^2 in A_2c^2 and A_3c^3 in writing down equations (13) and (14), it will be more correct to reject terms containing σ^2 in (16) and (17).

$\therefore$ From (2), we have, at any point (r, θ) of the vortex filament,

$$\omega=\frac{k}{2\pi a^2}\left[1+\frac{r^2}{c^2}\left(\frac{36\lambda+25}{16}\right)\cos 2\theta+\frac{r^3}{c^3}\,\frac{33\lambda+15}{16}\cos 3\theta\ldots\right] \quad \ldots \quad (18)$$

Here, we have found verticity correct to σ^2. The above method of treatment may be extended to find ω correct to higher powers of σ.

4. From (15) the velocity to a first approximation is given by

$$V=\frac{k}{16\pi c}(4\lambda+5)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right)$$

This is identical with the velocity of translation[1] of a ring of variable section whose vorticity is constant. The result might have been expected, inasmuchas if we neglect σ^2 and higher powers, the vorticity is found from (18) to be

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[1] *See* result (27) *Bull. Cal. Math. Soc.*, p. 127, Vol. 13.

$$\omega = \frac{k}{2\pi a^2} = \text{constant over the cross section.}$$

Hence, the velocity of translation must be same as that of a ring with constant vorticity at least to this order of approximation.

4

NOTE ON THE CONVERGENCE OF FOURIER'S SERIES.

The criteria of convergence of Fourier's series have been studied among others, by Dini, Jordan and De la Valleé Poussin; and certain isolated conditions (which are sufficient but not necessary) have been suggested by them. The condition proposed by the last is the most general of all; the proof of its greater generality, however, is not given in his "Course d'Analyse" (Ed. 1922, Tome II). The following proof was obtained by the writer while preparing for the Tripos. The proof becomes so short by the use of the property that an indefinite integral is of bounded variation according to both Riemann and Lebesgue.

It is assumed that $f(x)$ and its absolute value are integrable, either in the sense of Riemann or Lebesgue. We have,

$$\phi(\theta)=f(x+\theta)+f(x-\theta)-2s, \text{ where } s \text{ is properly chosen.}$$

I. *Dini's condition.*

If $\frac{|\phi(\theta)|}{\theta}$ is the integrable in the neighbourhood of '0', the Fourier's series of $f(x)$ converge towards $f(x)$. Here $s=f(x)$.

II. *Jordan's condition.*

The Fourier's series of $f(x)$ converge to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in the neighbourhood of which $f(x)$ is of bounded variation.

Here, $s=\frac{1}{2}[f(x+0)+f(x-0)]$ at all points of discontinuity of the first kind, $f(x+0), f(x-0)$ being equal to $f(x)$ at all points of regularity.

III. *De la Valleé Poussin's condition.*

The Fourier's series of $f(x)$ converge to s

where

$$\Phi_1(a)=\frac{1}{a}\int_0^a \phi(\theta)d\theta$$

is of bounded variation in the neighbourhood of '0'; s being so chosen that $\Phi_1(a)\longrightarrow 0$ as $a\longrightarrow 0$.

Proof.

A. By Dini's condition

since $\int_0^a \left|\frac{\phi(\theta)}{\theta}\right| d\theta$ exists, a being small; $\lambda(a) = \int_0^a \frac{\phi(\theta)}{\theta} d\theta$ does,

and $\lambda'(a) = \frac{\phi(a)}{a}$.

Now

$\Phi_1(a) = \frac{1}{a}\int_0^a \phi(\theta)d\theta = \frac{1}{a}\int_0^a \theta\lambda'(\theta)d\theta$ is of bounded variation by the *property of an indefinite integral* provided $\Phi_1(0)=0$.

$$\Phi_1(a) = \frac{1}{a}\ [a\lambda(a) - \int_0^a \lambda(\theta)d\theta] \lessgtr \lambda(a) - \lambda(a_1).$$

where $\lambda(a_1)$ is max $\lambda(a)$ in $0 \lessgtr a_1 \lessgtr a$.

But the expression on the right hand side $\longrightarrow 0$ with a, by Dini's condition

$$\therefore\ \Phi_1(0) = 0.$$

Thus if Dini's condition is satisfied, De la Valleé Poussin's is also satisfied.

B. By Jordan's condition $\Phi(a) \longrightarrow 0$ with a and is continuous near a.

$\therefore$ by the property of an *indefinite* integral $\int_0^a \phi(a)d\theta$ is of bounded variation and consequently $\Phi_1(a)$.

The only step to prove is that $\Phi_1(a) \longrightarrow 0$ with a.

Now,

$$\lim \Phi_1(a) = \lim_{a \longrightarrow 0} \frac{\Phi(a)-\Phi(0)}{\theta}, \qquad \text{where } \Phi(a)=\int_0^a \phi(\theta)d\theta$$

$$=\Phi'(0)=\phi(0), \qquad \text{by Jordan's condition.}$$

∴ if Jordan's condition is satisfied so is De la Valleé Poussin's.

K. C. D.

REVIEW

Wahrscheinlickeitsrechnung (I and II Vol.) von Prof. Dr. Otto Knopf: We have received two tiny volumes on the calculus of probabilities published by Walter de Gruyter and Co., of Berlin. They give in brief outline, besides the principles and methods of the calculus, several illustrative applications to insurance, meteorology, the theory of errors etc. They will, we trust, be welcome to all who would be content with a working knowledge of the principles or would have a rapid view of the whole before launching into a detailed study.